Control of exits from a safe region: a stochastic Melnikov approach

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Abstract

For a wide class of stochastically excited multistable systems the Melnikov necessary condition for chaos contains information on the effectiveness of the various frequency components of the excitation in inducing exits from a potential well. This information can be used to develop an open-loop control procedure aimed at reducing the system's exit rate. The Melnikov approach is based on a first-order approximation and is in principle applicable only for small perturbations. Nevertheless, results of simulations show that it is valid qualitatively even if the excitations are relatively large. We test the effectiveness of the proposed control procedure, and briefly review the use of the phase space flux factor to assess or design the control systems.

1 Introduction

For stochastically excited planar systems the stochastic Melnikov approach can provide a simple and effective basis for open-loop control aimed at reducing the rate of exit from a safe region of the phase space [1]-[8]. We briefly review relevant results of Melnikov theory, which in principle is applicable only for asymptotically small perturbations, and present results of simulations which show that in fact it is valid qualitatively even if the perturbations are relatively large. We describe the proposed control procedure and test its effectiveness by numerical simulations. We also review the use of the flux factor as an analytical tool that can help to assess or design Melnikov-based controls.

2 Melnikov necessary condition for exits from a potential well

We consider systems described by the equation

$$\ddot{x} = -\frac{dV}{dx} + \epsilon[\sigma G(t) - \beta \dot{x}] , \tag{1}$$

where $0 < \epsilon \ll 1$, $\sigma > 0$, $\beta > 0$ and $V(x)$ is a multiple-well potential. For the case of quasiperiodic excitation

$$G(t) = \sum_{n=1}^{N} \gamma_n \cos(\omega_n t + \phi_n) , \tag{2}$$

the Melnikov necessary condition for chaos is that the system's Melnikov function has simple zeros, i.e.,

$$-\beta k + \sigma \sum_{n=1}^{N} \gamma_n S(\omega_n) > 0 \tag{3}$$

where the constant k and the Melnikov scale factor $S(\omega)$ depend both on the potential $V(x)$ through the unperturbed system's homoclinic or heteroclinic orbits [9]. For a multistable system, Eq. (3) is also a necessary condition for exits from a well. From Eq. (3) it follows that the contribution to the promotion of chaos – and exits – by the component with amplitude γ_n depends upon $S(\omega_n)$, i.e., it is larger if $S(\omega_n)$ is large, and conversely.

If $G(t)$ is a stochastic process with unit variance and spectral density $2\pi\Psi_0(\omega)$, then over any finite time interval, however large, it can be approximated as closely as desired by the sum [10]

$$G_N(t) = \sum_{n=1}^{N} \gamma_n \cos(\omega_n t + \phi_n) \tag{4}$$

where $\gamma_n = \sqrt{2\pi\Psi(\omega_n)\Delta\omega}$, $\Delta\omega = \omega_c/N$, $\omega_n = n\Delta\omega$, ω_c is the cutoff frequency, ϕ_n is a random variable uniformly distributed in $[0, 2\pi]$, and N is finite. The Melnikov necessary condition for chaos can then be approximated by an expression with the same form as Eq. (3). It follows that the process $G(t)$ is more effective in promoting exits if its spectral density is concentrated at or near the frequency of the Melnikov scale factor's peak. However, we need to check whether this remains true if ϵ is relatively large.

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3 Dependence of exit rate upon frequency of added excitation

For the sake of transparency we consider the simple case $\sigma G(t) = \gamma_1 \cos \omega_1 t + \gamma_2 \cos \omega_2 t$. We refer to the first and second term of this sum as the basic and added excitation, respectively. We fix γ_1 and ω_1 and, for each of three different amplitudes γ_2, vary the frequency ω_2. We assume the Duffing-Holmes potential $V(x) = -ax^2/2 + bx^4/4$ in which $a = b = 0.5$, $\epsilon\beta = 0.045$, $\epsilon\gamma_1 = 0.114558$, and the three values $\gamma_2 = 0.0005$, $\gamma_2 = 0.004$, and $\gamma_2 = 0.032$. In the absence of the added excitation $\gamma_2 \cos \omega_2 t$, simulations yield a mean exit time $\tau_e \approx 60$. The mean exit time due to the effect of both the basic and added excitations is shown in Fig. 1 as a function of the frequency ω_2. Note that, even for relatively large total excitation, the Melnikov scale factor $S(\omega) = 2\pi\omega \text{sech}(\pi\omega/\sqrt{2})$ provides qualitative information on the dependence of the mean exit time upon the frequency of the added excitation: for frequencies ω_2 close to (far from) the frequency of $S(\omega)$'s peak the added excitation's effectiveness in promoting exits is strongest (weakest).

4 System stabilization by open - loop control: numerical simulations

For any given system, increasing the mean exit time τ_e by using an open-loop control approach can be achieved by adding to the excitation $\epsilon\sigma G(t)$ an appropriately chosen control force $\epsilon\sigma_c G_c(t)$, where σ_c has the same sign as σ and $|\sigma_c| < |\sigma|$. A trivial choice would be $G_c(t) = -G(t)$. The ratio between the average powers of the control force and the excitation force is then $Q = \sigma_c^2/\sigma^2$. We seek to use information contained in the Melnikov scale factor $S(\omega)$ to obtain control forces that would achieve mean exit time increases comparable to those achieved by the trivial control force, but more efficiently, that is, with a smaller power ratio Q.

To this end, instead of using the process $G_c(t) = G(t)$, we can apply a control force obtained from $G(t)$ by filtering out from this process the *ineffective components*, i.e., the components with frequencies for which $S(\omega)$ is small. In theory we can effect such filtering through multiplication of the spectral density of $G(t)$ by a unit rectangular function $U(\omega_1, \omega_2) \equiv H(\omega_1) - H(\omega_2)$, where $[\omega_1, \omega_2]$ is the frequency interval outside which the excitation components are ineffective, and H is the Heaviside function. In practice the control force has a time lag l with respect to the excitation, and practical filter characteristics differ from those associated with the unit rectangular function. Nevertheless the effect of filtering out ineffective components is of interest, and we show results of simulations illustrating that effect.

We considered Eq. (1) with the Duffing-Holmes potential in which $a = b = 1$ and parameters $\epsilon = 0.1$, $\beta = 0.45$. We assumed $2\pi\Psi_0(\omega) = 2\pi/5$ for $0 < \omega < 5$ and $2\pi\Psi_0(\omega) = 0$ otherwise. We first estimated the mean exit rate for the uncontrolled system. We then estimated mean exit rates for the system with four types of control force. The first type, denoted (a), is the trivial control force modified by introducing a time lag, $-\epsilon\sigma_c G(t - l)$. The second type, denoted (b), is obtained by passing the process $-\epsilon\sigma_c G(t - l)$ through an ideal filter that suppresses the ineffective components and leaves the other components unchanged. The third type, denoted (c), is obtained by passing $-\epsilon\sigma_c G(t - l)$ through the filter with the impulse response function of Fig. 2 ($A = 0.1$, $B = 2.25$). The fourth type, denoted (d), was obtained by passing $-\epsilon\sigma_c G(t - l)$ through the same filter as force (c), and then suppressing from the output the ineffective components while leaving the other components unchanged. The time lag was $l = 0.1$ in all cases. Figure 3 shows the spectrum of $G(t)$. It is seen that the Melnikov scale factor largely suppresses components with frequencies outside the interval $[\omega_1, \omega_2]$, where $\omega_1 = 0.2$ and $\omega_2 = 2.0$. These values were used to define the ineffective components. The ratios σ_c/σ were chosen so that the average power be equal for forces (a) and (b), and for forces (c) and (d). For forces (b) and (d) we assumed $\sigma_c/\sigma = 0.5$. The equal average power criterion yielded $\gamma_c/\sigma = 0.292$ for force (a) and $\gamma_c/\sigma = 0.347$ for force (c). The power ratios are then $Q = 0.2922 = 0.085$ for forces (a) and (b), and $Q = 0.108$ for forces (c) and (d).

The results of the simulations are shown in Fig. 4. Control force (a) (i.e., the trivial control with time lag l and $Q = 0.085$) reduces the exit rate by about 30 percent. For the same power ratio, control force (b), which uses information inherent in the Melnikov scale factor, performs significantly better, especially for smaller excitations, which are of interest in situations involving exits from a safe state. Similarly, for equal power ratios, force (d), obtained by eliminating the ineffective frequency components of force (c), is more effective than force (c).

It can be easily seen that the proposed open-loop control procedure is more effective for excitations with large ratio between the power of the ineffective components and the total excitation power. This is illustrated by examples in [6]. However, suppression of ineffective spectral components is only one requirement for an effective control filter. A second requirement is that the excitation be countered by control force components that have not only appropriate frequency content, but appropriate phase angles as well. An effective filter should satisfy both requirements. In the next section we discuss the use of the phase space flux factor as a tool for assessing or designing filters used to obtain the control force.

5 Flux-factor based open-loop control

To first order, the phase space flux factor is proportional to the time average of the area bounded by the time axis and the positive ordinate of the Melnikov process in any given time slice [2]. We denote by $\gamma M(t)$ and $\gamma_c M_c(t)$ the zero-mean fluctuating parts of the Melnikov processes induced by the excitation $\epsilon\gamma G(t)$ and the control force $\epsilon\gamma_c G_c(t)$, respectively,. The flux factor is defined as

$$\Phi \doteq E[(\gamma M(t) - \gamma_c M_c(t) - k)^+] \,. \qquad (5)$$

Define the random variable $\alpha Z(t) = \gamma M(t) - \gamma_c M_c(t)$, where $Z(t)$ has zero mean, unit variance, and cumulative distribution function $F_Z(z)$. The variance of the process $\alpha Z(t)$ is

$$\begin{aligned} \alpha^2 &= \mathrm{Var}[\gamma M(t) - \gamma_c M_c(t)] \\ &= \gamma^2 \alpha_M^2 + \gamma_c^2 \alpha_{M_c}^2 - 2\gamma\gamma_c \alpha_{MM_c} \end{aligned} \qquad (6)$$

where α_M^2, $\alpha_{M_c}^2$ and α_{MM_c} are the variances and the covariance of the processes $M(t)$ and $M_c(t)$. The flux factor can be written

$$\begin{aligned} \Phi = E[(\alpha Z - k)^+] &= \int_{k/\alpha}^{\infty} (\alpha z - k) dF_Z(z) \\ &= \alpha \int_{k/\alpha}^{\infty} [1 - F_Z(z)] dz \,. \end{aligned} \qquad (7)$$

The last result of Eq. (7) is obtained by integrating by parts the integral $\int z dF_Z(z)$, and is written as

$$\Phi = k\mu(\alpha/k) \,, \mu(x) \equiv x \int_{k/\alpha}^{\infty} [1 - F_Z(z)] dz. \qquad (8)$$

In the particular case of Gaussian excitation and control force, Z is also Gaussian. It can be verified [4] that, for $\alpha = 0$, $\Phi = 0$, and if the probability density function $f_Z(z)$ decreases faster than z^{-3} as $z \to \infty$, $d\Phi/d\alpha = 0$ and $d^2\Phi/d^2\alpha = 0$. For $\alpha > 0$, $d\Phi/d\alpha > 0$ and $d^2\Phi/d^2\alpha > 0$. Since Φ is a monotonically decreasing function of α the flux factor is smaller for the controlled system than for the system with no control force ($\gamma_c = 0$) if and only if $\alpha < \gamma\alpha_M$ or, from Eq. (6),

$$\gamma_c/\gamma < 2\alpha_{MM_c}/\alpha_{M_c}^2 \,. \qquad (9)$$

The optimal ratio γ_c/γ - the ratio which, given $M(t)$ and $M_c(t)$, minimizes α^2 - is

$$\gamma_{c,opt}/\gamma = \alpha_{MM_c}/\alpha_{M_c}^2 \,. \qquad (10)$$

The flux factor as reduced by the action of the control force with optimal strength $\gamma_{c,opt}$ is

$$\Phi_{r,o} = k\mu(R\gamma\alpha_{M_c}/k) \,, R = \sqrt{1 - \rho_{MM_c}^2} \qquad (11)$$

where $0 < \rho_{MM_c} = \alpha_{MM_c}/(\alpha_M \alpha_{M_c}) \leq 1$ is the control force's flux reduction index. The smaller the index R, the more effective is the control force. Approximate expressions for R as functions of the system potential $V(x)$, the spectral density of the excitation, the time lag l, and the filter gain and phase, were derived from Eq. (12) in [4]. These expressions can be used for the assessment of any given filter design.

Let $F(t)$ denote the process obtained by passing the excitation process through a measurement filter and a lag filter that causes the output of the measurement filter to experience a lag l. Given the system's potential $V(x)$ (or, equivalently, the system's Melnikov scale factor), passing $F(t)$ through a control filter yields a control force that can be optimized in the sense that it will achieve the control objective of producing the smallest possible flux factor. If the available control power exceeds a certain threshold the optimal control filter is a Wiener filter; otherwise, a non-Wiener solution is applicable [5].

6 Conclusions

A Melnikov-based open-loop approach to the control of a wide class of nonlinear stochastic systems was described. Exploratory numerical simulations and investigations on the reduction of the flux factor by a control force indicate that the information contained in the Melnikov scale factor can help achieve a relatively efficient stabilization of the system. The degree to which this is the case depends on the system (i.e., on its Melnikov scale factor), the excitation spectrum, and the design of the control filter.

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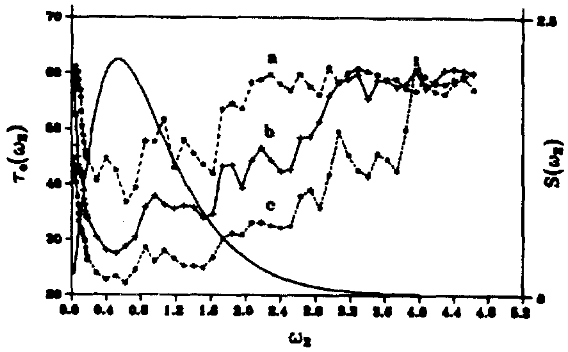

Figure 1 Dependence of mean escape time τ_e on frequency of added excitation for amplitudes: (a) $\gamma_2 = 0.0005$, (b) $\gamma_2 = 0.004$, (c) $\gamma_2 = 0.032$. Solid line: Melnikov scale factor.

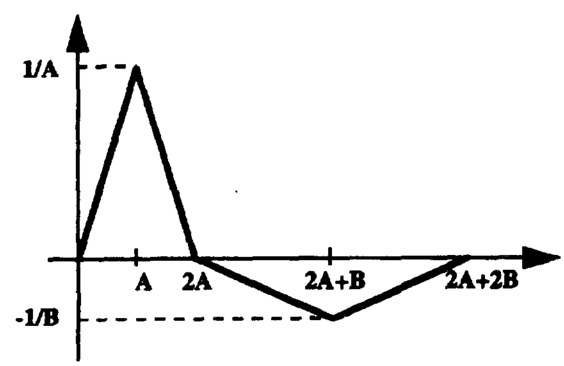

Figure 2 Impulse response function of two-parameter filter with initial response and recoil.

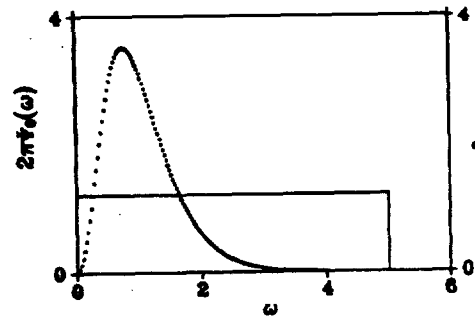

Figure 3 Spectral density of excitation (solid line) and square of Melnikov scale factor (interrupted line).

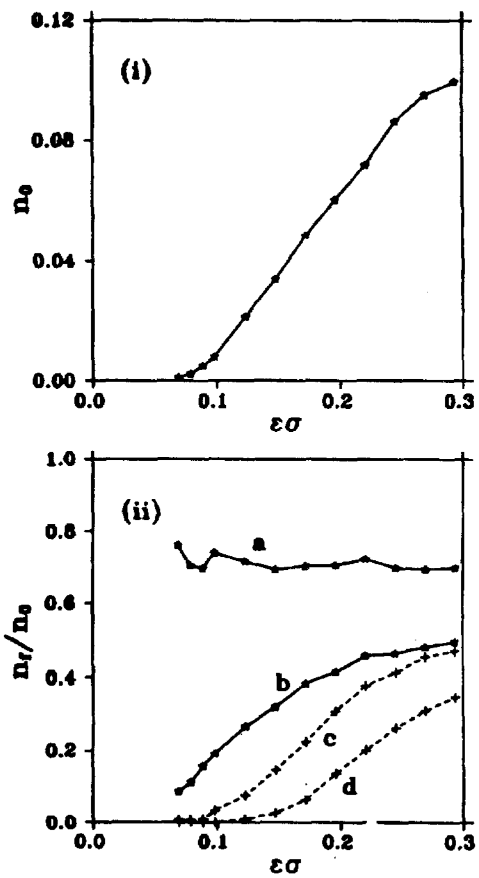

Figure 4 (i) Escape rate for uncontrolled system; (ii) ratio n_f/n_0 between escape rates of system with control forces *a, b, c, d* and escape rate of uncontrolled system.