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Noise-Induced Sensitivity to Initial Conditions

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ABSTRACT Deterministic chaos and noise-induced basin hopping are closely related in a broad class of multistable dynamical systems. A necessary condition for sensitivity to initial conditions, based on the generalized Melnikov function and originally derived for deterministic systems, can be extended to systems excited by noise. This extension involves the representation of noise processes as sums of terms with random parameters. Gaussian noise and shot noise can be accommodated for both additive and multiplicative excitations. Our extension of the Melnikov approach shows that, for the class of noise-excited systems being considered, basin hopping implies sensitivity to initial conditions. Applications of this approach to noise-excited systems are discussed.

6.1 Introduction

Multistable systems excited by noise can exhibit irregular motions with jumps between regions associated with the basins of attraction of their noise-free counterparts. Such behavior has been referred to as *basin hopping*, or *stochastic motion with jumps*. The same systems can have irregular motions with jumps in the absence of noise. The term *deterministic chaos* is used in this case. Numerical simulations and physical experiments in various fields (e.g., physics [1], chemistry [2], biomedicine [3], fluid elasticity [4]) have shown that deterministic and stochastic motions with jumps can be visually indistinguishable (Fig. 6.1). Moreover, it has been shown recently that features previously believed to characterize deterministic chaos can be present in stochastic systems as well. These features include a finite, predictable value of the correlation dimension [5], a convergent K_2 entropy [6], an exponential falloff of the power spectrum [7], and a positive largest Lyapounov exponent [8].

For weakly perturbed multistable systems whose unperturbed counterparts have homoclinic or heteroclinic orbits, a useful mathematical correspondence exists between deterministic chaos and stochastic motion with jumps. To show this, we represent Gaussian noise processes as sums of harmonic terms with random amplitudes, frequencies, and phase angles. Similar techniques are used for other types of noise. Two concepts, the generalized Melnikov function (GMF) and phase space flux, originally developed for deterministic systems, can then be extended to systems excited by additive and/or multiplicative noise. This extension has a

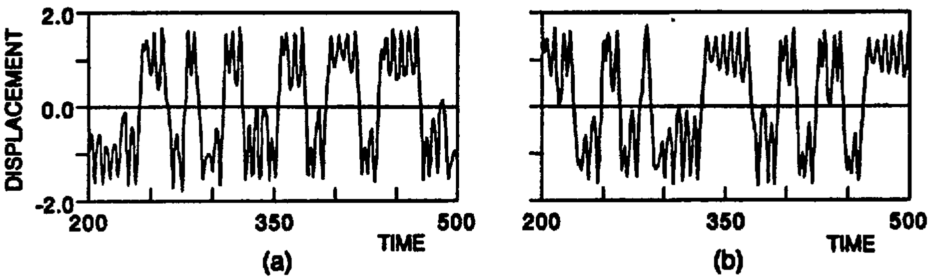

FIGURE 6.1. Dynamics of an oscillator. (a) Periodic forcing. (b) Noise driven.

theoretical consequence: For weakly perturbed multistable systems, noise-induced jumps out of regions associated with the unperturbed system's potential wells imply sensitivity to initial conditions (SIC). The practical consequence of this extension is that methods for investigating and characterizing the behavior of potentially chaotic deterministic systems can also be used to investigate systems subjected to noise.

In Sec. 6.2, we review results obtained for one-degree-of-freedom systems. We consider systems excited additively or multiplicatively by Gaussian noise and other types of noise. Although our approach does not at present allow mean exit times (times between jumps) to be estimated, it does establish certain lower bounds on exit time probabilities. In addition, for tail-limited noise and dichotomic noise, criteria can be obtained that guarantee that exits from a safe region associated with a potential well cannot occur. In Sec. 6.3, we verify for a specific system that results based on the Melnikov approach coincide with results based on the use of the Fokker-Planck equation. Section 6.4 discusses the extension of results in Sec. 6.2 to higher-dimensional and spatially extended systems. Section 6.5 summarizes our conclusions.

6.2 One-Degree-of-Freedom Systems

6.2.1 Dynamical Systems and the GMF

We now demonstrate the basic ideas of our approach for systems of the form

$$\ddot{z} = -V'(z) + \epsilon[g(t) + \rho f(z, \dot{z})G(t) - \beta\dot{z}], \tag{6.1}$$

where z is the state of the system, V is an energy potential, $0 \leq \epsilon \ll 1$, ρ and β are nonnegative, g is a smooth function representing deterministic forcing, G is a random process that models the stochastic forcing, and $f(z, \dot{z})$ is a smooth function defining the excitation dependence on the system state. When f is constant, the random excitation is additive; otherwise, the excitation is multiplicative.

We assume that the unperturbed ($\epsilon = 0$) counterpart of the system in Eq. (6.1) has two hyperbolic fixed points connected by a heteroclinic orbit $(z_s(t), \dot{z}_s(t))$. If the two fixed points coincide, then the orbit $(z_s(t), \dot{z}_s(t))$ is homoclinic. These assumptions cover a large number of systems of interest in applications. The GMF is, formally,

$$M(t) = -\int_{-\infty}^{\infty} \dot{z}_s^2(\tau)d\tau + \int_{-\infty}^{\infty} \dot{z}_s(\tau)g(\tau + t)d\tau + \int_{-\infty}^{\infty} \dot{z}_s(\tau)G(\tau + t)d\tau. \quad (6.2)$$

We define orbit filters $\mathcal{F}$ and $\mathcal{F}_1$ with respective impulse responses

$$h(t) = \dot{z}_s(-t), \quad h_1(t) = \dot{z}_s(-t)f(z_s(-t), \dot{z}_s(-t)). \quad (6.3)$$

Then

$$M[g, G] = -I\beta + \mathcal{F}[g] + \mathcal{F}_1[G], \quad I = \int_{-\infty}^{\infty} \dot{z}_s^2(t)dt. \quad (6.4)$$

For systems with additive noise, f is constant and $\mathcal{F}_1 = \mathcal{F}$.

We consider first the case of deterministic and quasiperiodic excitation, where $\rho = 0$ and

$$g(t) = \sum_{i=1}^{m} \gamma_i \cos(\omega_i t + \phi_i). \quad (6.5)$$

The GMF is then

$$M(t) = -I\beta + \mathcal{F}[\sum_{i=1}^{m} \gamma_i \cos(\omega_i t + \phi_i)]. \quad (6.6)$$

The GMF is a measure of the distance separating the stable and unstable manifolds of the hyperbolic fixed point. If the GMF has simple zeros, those manifolds intersect transversely. These intersections form an infinity of lobes, giving rise to structures topologically equivalent to the Smale horseshoe map, which may lead to chaotic behavior [9]. Thus, if the system parameters belong to the range of parameters for which the GMF exhibits simple zeros, deterministic chaos is possible.

The formal expression of the GMF is mathematically justified when the noise process G is uniformly bounded and ensemble-uniformly continuous (EUC) [8].

6.2.2 *Additive Gaussian Noise*

As an example, we consider the case of an additive Gaussian excitation. We approximate the Gaussian excitation using a randomly weighted modification of Shinozuka noise:

$$G(t) = \sqrt{\frac{2}{N}} \sum_{n=m+1}^{m+N} \frac{\sigma \cos(\omega_n t + \phi_n)}{S(\omega_n)}, \quad \sigma^2 = \int_0^{\infty} S^2(\omega)\Psi(\omega)d\omega, \quad (6.7)$$

where ω_n, ϕ_n, $n = m + 1, \ldots, m + N$ are independent random variables, ω_n, $n = m + 1, \ldots, n = m + N$ are nonnegative with common distribution $\Psi_o(\omega) =$

$S^2(\omega)\Psi(\omega)/\sigma^2$, ϕ_n are identically uniformly distributed over $[0, 2\pi]$, Ψ is the spectral density of G, and S is the modulus of the Fourier transform of $\dot{z}_s(-t)$. In the special case of white noise, $\Psi(\omega) = 1$. In this case, using Parseval's theorem,

$$\sigma^2 = \int_0^{\infty} S^2(\omega)\Psi(\omega)d\omega = \frac{1}{2}\int_{-\infty}^{\infty} S^2(\omega)\Psi(\omega)d\omega = \pi \int_{-\infty}^{\infty} \dot{z}_s^2(-t)dt = \pi I.$$

The noise process G in Eq. (6.7) is uniformly bounded and EUC with zero mean and unit variance. G can be specified with any desired power spectrum and is Gaussian in the limit as $N \to \infty$ [8, 10, 11, 12, 13]. Realizations of this noise model are difficult to distinguish from realizations of band-limited Gaussian noise [8].

Given G of the form in Eq. (6.7) we obtain an ergodic ensemble of GMFs:

$$M(t) = -I\beta + \mathcal{F}[\sum_{i=1}^{m} \gamma_i \cos(\omega_i t + \phi_i)] + \rho\sigma\sqrt{\frac{2}{N}}\mathcal{F}_1[\sum_{i=m+1}^{m+N} \gamma_i \cos(\omega_i t + \phi_i)]. \quad (6.8)$$

The necessary condition for SIC for any particular realization of the noise is that $M(t)$ have simple zeros. In the Gaussian limit, this condition is satisfied with probability one in the presence of even vanishingly small noise, $\rho \to 0$. We thus recover the well-known result that jumps between stable states must eventually occur for any realization of the Gaussian excitation, however small, although the time between jumps can be quite long. We have also established that, given system parameters in the necessary ranges, noise-induced motion with jumps is SIC and interpretable as chaotic motion on a single strange attractor [8, 9].

6.2.3 *Other Forms of Noise*

Similar types of results are possible for forms of non-Gaussian noise. An oscillator subject to multiplicative shot noise is considered by Frey and Simiu [14], in which shot noise is represented by a uniformly bounded, EUC random process.

Non-Gaussian noise processes with infinite tails and specified spectral density can also be treated. Representation of such processes as ensembles of harmonic sums is achieved by generating Gaussian paths and then mapping these paths into non-Gaussian paths through an iterative procedure aimed at matching the target spectrum. General conditions for the convergence of this procedure remain to be established. A successful illustration of the method is available in a paper by Yamazaki and Shinozuka [13]. When convergent, the method gives results similar to Eq. (6.8).

In multistable systems, each potential well has a corresponding GMF. These GMFs reflect the differences, if any, in the shape and depth of the wells. They will also reflect the dependence of multiplicative noise on state as, for example, in the blowtorch theorem [15]. This suggests that appropriate GMF ensemble statistics may possibly be used as indicators of relative stability.

6.2.4 Average Flux Factor

The phase-space flux is the rate of phase-space transport across the pseudoseparatrix via lobes defined by intersecting stable and unstable manifolds. Phase-space flux is a measure of the degree to which the system dynamics are chaotic [9]. For small ϵ, the average phase-space flux is $\epsilon\Phi + \mathcal{O}(\epsilon^2)$, where the flux factor Φ is proportional to a time average of the GMF [9, 16].

Frey and Simiu [8] showed that, for the system in Eq. (6.1) with additive noise G, the flux factor is nonrandom with the expression

$$\Phi = E[(\rho A + B - I\beta)^{+}], \tag{6.9}$$

where A is a random variable with distribution equal to the marginal distribution of the process $\mathcal{F}[G]$, B is a random variable with distribution equal to the stationary mean distribution of the function $\mathcal{F}[g]$, A and B are independent, and $+$ denotes positive part of. When G is Gaussian (or nearly so, as in Eq. (6.7)), A is a Gaussian (or nearly Gaussian) random variable. Similar expressions were derived by Frey and Simiu [14] for systems with multiplicative Gaussian noise and multiplicative shot noise. Our earlier comment on the possible use of GMF statistics as indicators of relative stability applies as well to the average flux factor. We note that unlike the GMF, the notion of phase-space flux appears so far to be applicable only to systems with one degree of freedom.

6.2.5 Probability of Exit from a Safe Region

The reliability of a system is frequently expressed in terms of the probability of exit from a safe region during a time interval $\mathcal{I}$ of duration T. For simplicity, assume that no deterministic forcing is present, $g = 0$. During the interval $\mathcal{I}$, some realizations of $M(t)$ will have simple zeros while others will not. In the absence of simple zeros, the stable and unstable manifolds do not intersect, and escape from within the pseudoseparatrix is impossible. In the Gaussian limit $N \to \infty$ and, provided that $I\beta/\rho\sigma$ is sufficiently large (greater than 3, say), the probability that $M(t)$ will not have simple zeros during this interval is [17]

$$\mathrm{Prob}\{M(t) < 0\} = \exp[-E(k)T], \tag{6.10}$$

$$E(k) = \mu \exp\left[-\frac{I\beta}{2\rho\sigma}\right], \quad \mu = \frac{1}{4\pi^2\sigma^2}\int_{-\infty}^{\infty} S^2(\omega)\omega^2\Psi(\omega)d\omega. \tag{6.11}$$

We denote by t_{ex} the time between successive jumps across a boundary separating regions of phase space associated with the system's potential wells. (Note that for the perturbed system these boundaries, referred to as pseudoseparatrices, differ from the separatrices of the unperturbed system). Since t_{ex} must be larger than T (there can be no exit as long as the stable and unstable manifolds do not intersect), Eq. (6.10) provides a lower bound for the probability that $t_{ex} > T$. A similar result also holds for $g \neq 0$.

Results of this type can also be obtained for non-Gaussian noise excitation. For noise with tail-limited distributions—a case of interest in a number of engineering applications [18]—the distribution of the GMF is also tail limited. It is then possible to derive, in addition to probability bounds, criteria guaranteeing the absence of exits (and SIC). This is of particular interest in the case of dichotomic noise.

6.3 Mean Time Between Peaks—Brundsen-Holmes Oscillator

In this section, we briefly review an application of our Melnikov-based approach for which the result can be compared with a result obtained through the use of the Fokker-Planck equation [12, 19]. The oscillator is defined by Eq. (6.1), where $V(z) = z^3 - z$, $\gamma = 0$, G is Gaussian white noise, and β, instead of denoting constant damping, is a function $\beta = \delta - kz^2$ of the system state z. The mean time between successive maxima of $|z(t)|$ is [19]

$$T = K - \frac{\ln(\overline{M})}{\lambda_u}, \tag{6.12}$$

where K is a constant, λ_u is the eigenvalue associated with the unstable manifold linearized about the saddle point, and $\overline{M}$ is proportional to the average Melnikov distance separating the stable and unstable manifolds of the perturbed system. The following result was obtained in Stone and Holmes [19]:

$$T = K_1 - \frac{\ln(\epsilon\rho)}{\lambda_u}. \tag{6.13}$$

For forcing given by Eq. (6.7), it follows from the definition of the GMF that $\overline{M}$ has the same distribution as the ensemble average of the modulus of the Melnikov function at time $t = 0$ [12]. In the white noise limit, this yields

$$\overline{M} = \frac{\rho\sigma}{\sqrt{2\pi}} \int_0^{\infty} |y| e^{-y^2/2} dy = \sqrt{\frac{2}{\pi}} \rho\sigma. \tag{6.14}$$

Substitution of Eq. (6.14) into Eq. (6.12) yields Eq. (6.13); that is, our Melnikov-based approach and the application of the Fokker-Planck equation in Stone and Holmes [19] yield the same result.

6.4 Higher-Degree-of-Freedom Systems

6.4.1 *Slowly Varying Oscillators*

The GMF can also be used to obtain a necessary condition for noise-induced SIC in systems of the form

$$\dot{x} = \frac{\partial}{\partial y} H(x, y, z) + \epsilon[g_1(x, y, z, t; \mu) + \rho_1 G_1(t)], \tag{6.15}$$

$$\dot{y} = -\frac{\partial}{\partial x}H(x, y, z) + \epsilon[g_2(x, y, z, t; \mu) + \rho_2 G_2(t)], \tag{6.16}$$

$$\dot{z} = \epsilon g_3(x, y, z, t; \mu) + \epsilon\rho_3 G_3(t), \tag{6.17}$$

where $0 \leq \epsilon \ll 1$, H is a Hamiltonian with a vector z of parameters, μ is a vector of parameters, and the functions g_i, $i = 1, 2, 3$ are quasiperiodic with m_i-incommensurate frequencies ω_{ij}, $j = 1, \ldots, m_i$, $i = 1, 2, 3$. If there exists an open interval $J \subset \mathcal{R}$ such that, for every $z \in J$, the unperturbed system ($\epsilon = 0$) has a hyperbolic fixed point p_z with a homoclinic solution $q_z(t)$ connecting p_z to itself, then the fixed points form a smooth curve $C = \gamma(z)$ in the phase space x, y, z [20]. The homoclinic solutions form a two-dimensional surface Γ, which connects C to itself.

Let $G_i(t)$, $i = 1, 2, 3$ be independent Gaussian processes approximated by modified Shinozuka processes as in Eq. (6.7). A necessary condition for SIC is that the GMF for the system have simple zeros. The proof is similar to that given by Wiggins and Holmes [20] for periodic forcing. The expression for the GMF has the same form as in the case of periodic excitations [20] except that (i) the quasiperiodic functions $g_i(\gamma(z_o)) + \rho_i G_i(\gamma(z_o))$, $i = 1, 2, 3$ are substituted for their periodic counterparts, and (ii) z_o is a point such that

$$\overline{g_3(\gamma(z_o)) + \rho_3 G_3(\gamma(z_o))} = 0, \tag{6.18}$$

$$\frac{d}{dz}\overline{g_3(\gamma(z_o)) + \rho_3 G_3(\gamma(z_o))} \neq 0, \tag{6.19}$$

where the overbar denotes averaging over time [21].

The Melnikov approach restricted to a periodic vector (g_1, g_2, g_3) of deterministic excitations has been applied [22] to the estimation of the onset of chaos induced by harmonic wind velocity fluctuations in ocean flow over certain topographies. Its extension to the case of quasiperiodic functions, including representations of Gaussian noise, allows the solution of the more realistic problem in which the wind fluctuations are random rather than periodic.

6.4.2 *A Spatially Extended System*

To demonstrate the possibility of extending our work to spatially extended systems, we consider snap-through motion (jumps) induced by transverse random excitation of a buckled column with continuously distributed mass. The excitation may be due, for example, to seismic motion, fluctuating hydrodynamic effects, aerodynamic turbulence, or effects arising in mechanical systems. Assuming uniform mechanical properties over the length of the column, the equation of motion of the column is

$$z_{tt} + z_{yyyy} + \Gamma z_{yy} - \xi z_{yy}\int_0^1 z_y^2(\zeta, t)d\zeta = \epsilon[\gamma(y)\cos(\omega t) + \rho G(t) - \beta z_t], \tag{6.20}$$

where $z(y, t)$ is the transverse deflection at time t and point y ($0 \leq y \leq 1$) along the length of the column. Γ is the external compression load, ξ is the stiffness due to membrane stress, β reflects the degree of damping, $\gamma(y)\cos(\omega t)$ is the deterministic forcing, and $\rho G(t)$ is the random excitation. Both ends of the column are assumed to be hinged so the boundary conditions on Eq. (6.20) are $z(0,t) = z(1,t) = z_{yy}(0,t) = z_{yy}(1,t) = 0$. The eigenvalues of the linearized, unforced equations are then $\lambda_j = \pm\pi j(\Gamma - \pi j^2)^{1/2}$, $j = 1, 2, \ldots$. As in Holmes and Marsden [23], we assume $\pi^2 < \Gamma < 4\pi^2$. Then the solution $z = 0$ has one positive and one negative eigenvalue, and the system with $\epsilon = 0$ and $\xi > 0$ has two nontrivial buckled equilibrium states.

For $\rho = 0$ the GMF of the column is

$$M(t) = \int_{-\infty}^{\infty}\int_0^1 [\gamma(y)\cos(\tau)\dot{z}_o(y,\tau - t) - \beta\dot{z}_o^2(y,\tau - t)]dyd\tau, \tag{6.21}$$

where $z_o(y,t) = \Gamma_1\sin(\pi y)\text{sech}(\pi\xi^{1/2}t)$ is the system coordinate along the homoclinic orbit of the unperturbed system, and $\Gamma_1 = 2[(\Gamma - \pi^2)/\xi]^{1/2}$. After inserting this expression for the homoclinic orbit into the GMF,

$$M(t) = \frac{\beta\Gamma_1^3\xi^{1/2}}{6\pi} - \frac{\omega\Gamma_1(\alpha/2 + 2\overline{\gamma}/\pi)\sin(\omega t)}{\pi\cosh(\omega/(\Gamma_1\xi^{1/2}))}, \tag{6.22}$$

where $\overline{\gamma}$ is the mean of $\gamma(y)$, and α is a coefficient in the Fourier expansion of $\gamma(y)$ [23]. Equations (6.21) and (6.22) are valid, provided the following nonresonance condition is satisfied:

$$j^2\pi^2(j^2\pi^2 - \Gamma) \neq \omega^2, \quad j = 2, 3, 4, \ldots. \tag{6.23}$$

By expanding $z(y,t)$ in the eigenfunctions $\sin(j\pi z)$ of the linearized problem and using Galerkin's method, the following equations are obtained for the modal coefficients $a_j(t)$, $j = 1, 2, \ldots$

$$a_j(1 + \epsilon\beta + j^2\pi^2(j^2\pi^2 - \Gamma)) = \epsilon\gamma_j\cos(\omega t), \quad \gamma_j = \int_0^1 \sin(j\pi y)dy. \tag{6.24}$$

If Eq. (6.23) is satisfied, $a_j(t)$ is $\mathcal{O}(\epsilon)$ for all j, and the Poincaré map of the perturbed map possesses a hyperbolic saddle point p_ϵ such that $p_\epsilon = p_0 + \mathcal{O}(\epsilon)$, where p_0 is the saddle point of the unperturbed system. If the nonresonance condition is not met, then the modal coefficients are $\mathcal{O}(1)$ and the GMF is not meaningful.

If $\gamma(y) = 0$, $\rho \neq 0$ and the spectrum of $G(t)$ is band limited, it follows from elementary random vibration theory that $a_j(t)$ is $\mathcal{O}(\epsilon^{1/2})$ [24, 25]. For sufficiently small ϵ, this is sufficient to ensure that the saddle point and its associated stable and unstable manifolds persist under perturbation. The expression for the GMF then has the same form as Eq. (6.21), except that $\gamma\cos(\omega t)$ is replaced by a modified Shinozuka sum of randomly weighted harmonics. The proof of this follows the same steps as in the work of Frey and Simiu [8] and Wiggins [16]. $G(t)$ can be superimposed on one or more harmonic excitations (which must satisfy the nonresonance condition) and need not be uniform or coherent over the column length.

6.5 Conclusions

Melnikov theory, originally developed for deterministic systems, can be extended to noise-excited multistable systems. This extension reveals that for weakly perturbed multistable systems, noise can induce sensitivity to initial conditions. This extension can also be used to establish lower bounds on probabilities of exit from a safe region during a specified time interval. For tail-limited noise including, for example, dichotomic noise, criteria can be given that provide a guarantee that exits will not occur. Our approach may also provide at least rudimentary measures of relative stability for systems with multiplicative noise. The range of applicability of our approach includes one-degree-of-freedom systems, higher-degree-of-freedom systems such as slowly varying oscillators, and spatially extended systems such as a buckled column with continuously distributed mass.

Our interest in Melnikov-based methods is motivated by applications to oceanography and structural/mechanical reliability, but we believe potential applications to other disciplines exist.

6.6 Acknowledgment

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