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A NEW TOOL FOR THE INVESTIGATION OF A CLASS OF NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS: THE MELNIKOV PROCESS

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1. Introduction

The Melnikov process, a construct rooted in chaotic dynamics theory, was recently developed as a tool for the investigation of a broad class of nonlinear stochastic differential equations [1-6]. This paper briefly reviews the stochastic Melnikov-based approach and applications to (i) oceanography, (ii) open-loop control of stochastic nonlinear systems, and (iii) snap-through of buckled beams with distributed mass and distributed random loading.

2. Melnikov Processes

2.1 DYNAMICAL SYSTEMS

The stochastic Melnikov-based approach reviewed in this paper is applicable, among others, to systems described by the equation

$$\ddot{z}=f(z)+\epsilon[\beta\dot{z}+\gamma G(t)] \tag{1}$$

where $|\epsilon|\ll 1$, and the function f and the random process $G(t)$ are assumed to be sufficiently smooth and bounded. (This restriction on $G(t)$ may be in practice be removed: if $G(t)$ is not smooth and/or bounded, it may be replaced by a bounded and sufficiently smooth process $G_a(t)$ that approximates $G(t)$ as closely as desired over any any arbitrarily large, though finite, time interval. For example, one may approximate broadband Gaussian noise by a sum of a large number of bounded harmonic terms with random frequencies and phases [7]; white noise by broadband noise with constant spectrum and very large cut-off frequency [4]; square wave coin-toss dichotomous noise by a series of sums with a random (coin-toss) parameter and a non-random parameter defining the position of the square wave on the time axis, each sum in the series representing an arbitrarily close Fourier approximation of a square wave [5].) In addition, it is assumed that the unperturbed system has two hyperbolic critical points z_+ and z_-, not necessarily distinct, and that there is an

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orbit $\gamma(t)$ which connects them. (The orbit is called homoclinic if the two points coincide, and heteroclinic otherwise.) In particular this assumption holds for systems with $f(z)=dV(z)/dz$, where $V(z)$ is a two- or multi-well potential. Examples, in addition to others given later in the paper, are the Duffing-Holmes equation, the Josephson junction, and models of vessel capsizing in beam seas [2].

Equation 1 with $\epsilon=0$ is referred to as the unperturbed system.

2.2 MELNIKOV-BASED NECESSARY CONDITONS FOR ESCAPES

2.2.1 *Persistence Theorem*

For the system approximating Eq. 1 [i.e., Eq. 1 in which $G_a(t)$ is substituted for $G(t)$], the persistence theorem can be used to show that, for sufficiently small ϵ, the hyperbolic critical points persist in a phase space slice through the approximating system's extended phase space [8]. This means that the perturbed system will possess separated stable and unstable manifolds.

2.2.2 *Application of Smale-Birkhoff Theorem*

The Smale-Birkhoff theorem states that a necessary condition for the occurrence of chaos (i.e., for sensitivity to initial conditions) is that the stable and unstable manifolds of system (1) intersect transversely [8]. Crossing of a potential barrier (i.e., escapes from regions of phase space associated with the interior of a potential well) can occur only via lobes resulting from the transverse intersections of the system's stable and unstable manifolds [8]. The existence of such intersections is the a necessary condition for chaotic transport. In particular, for systems with stochastic excitation, it is a necessary condition for the occurrence of escapes. Therein lies the connection between chaotic dynamics and the study of escapes in nonlinear multistable stochastic differential equations [1,4].

To first order, the Melnikov process is a measure of the distance beween the stable and unstable manifolds of the stochastic system approximating Eq. 1. If a realization of the Melnikov process has simple zeros, the corresponding stable and unstable manifolds intersect transversely.

It can be shown that: (a) the process $G_a(t)$ induces a Melnikov process arbitrarily close to the process

$$M(t) = -\beta\int_{-\infty}^{\infty}\dot{z}_s^2(\tau)d\tau+\gamma\int_{-\infty}^{\infty}h(\tau)G(t-\tau)d\tau \tag{2}$$

where $\dot{z}_s$ is the ordinate in the $z,\dot{z}$ phase plane of the unperturbed system's heteroclinic or homoclinic orbit; (b) the filter in the convolution integral of Eq. 2 is $h(t) = \dot{z}_s(-t)$; and (c) the mean zero upcrossing time of the Melnikov process, τ_u, is a lower bound for the system's mean escape time, τ_e [1,3,4]. To increase τ_e it is therefore necessary to increase τ_u. If the marginal distribution of the noise $G(t)$ is bounded, for values of β, γ such that $M(t) \leq 0$ for $-\infty < t < \infty$ no escapes can ever occur.

In some applications the mean value and spectral density of $M(t)$ are useful. They are, respectively, $-\beta K$, where K denotes the value of the first integral of Eq. 2, and

$$2\pi\Psi_M(\omega) = 2\pi\Psi(\omega)\gamma^2 S^2(\omega). \tag{3}$$

$S(\omega)$, the Fourier transform of $h(t)$, is known as the Melnikov relative scale factor.

Finally, we note that although in theory the results just reviewed are valid for small ϵ, in practice they were found to hold even for relatively large ϵ [9].

2.3 EXTENSIONS TO OTHER TYPES OF DYNAMICAL SYSTEMS

Results similar to those just reviewed have been extended to: systems with multiplicative noise [4]; a class of systems of two first-order stochastic differential equations with a slowly-varying parameter; and a stochastic partial differential equation whose deterministic counterpart was first studied by Marsden and Holmes [10] -- see Sections 3 to 5. Similar results can be obtained for multi-degree of freedom systems whose unperturbed counterparts have homo/heteroclinic orbits.

3. Model of Along-shore Currents Due to Randomly Fluctuating Winds Over Continental Shelf With Periodic Corrugations Normal to Coastline

This problem was studied by Allen et al. (1991) [11] for the case of harmonically fluctuating wind forcing. In the absence of friction and forcing this model exhibits homoclinic orbits due to the presence of ocean bottom corrugations normal to the coastline, which correspond in the mathematical model to potential wells separated by a barrier. Under excitation by wind with low frequency *harmonic* fluctuations, and in the presence of friction, for low wind speeds the steady-state motion of a fluid particle will occur within a well for all time. However, if the wind speeds are sufficiently strong, the system's Melnikov function will have simple zeros, and the particle can behave chaotically, that is, move back and forth across the potential barrier in an apparently random fashion [11].

The equations of motion governing the current motion belong to a class of deterministic second order systems with a slowly varying parameter studied by Wiggins and Holmes (1987) [12]. An extension to the stochastic case was reported in [3], and allowed consideration of the more realistic case where the alongshore currents are excited by random wind fluctuations. The fluctuations induce a Melnikov process (that is, an ensemble of Melnikov functions). Assume for example that the excitation is Gaussian. Then, with probability one, the Melnikov process will have simple zeros, and escapes across the potential barrier are possible -- provided that one waits a sufficiently long time. However, the probability that escapes will occur within a specified finite time interval is less than one. Using estimates of the mean zero upcrossing time of the Melnikov process, a weak upper bound for this probability was estimated in [3,4].

4. Melnikov-based Open-loop Control

The performance of certain nonlinear stochastic systems is deemed acceptable if, during a specified time interval, the systems have sufficiently low probabilities of escape from a region of phase space associated with a potential well. An open-loop control method for reducing these probabilities was proposed in [13]. The method is applicable to stochastic systems defined in Sections 2.1 and 2.3. The Melnikov relative scale factors, defined in Section 2.2, are system properties containing information on the frequencies of the random forcing spectral components that are most effective in inducing escapes. An ideal open-loop control force applied to the system would be equal to the negative of a fraction of the exciting force from which the ineffective components have been filtered out. Limitations inherent in any practical control system make it impossible to achieve such an ideal control. Nevertheless, numerical simulations summarized in this section show that, substantial advantages can be achieved in some cases by designing control systems that take into account the information contained in the Melnikov scale factors.

4.1 NUMERICAL SIMULATIONS

4.1.1 *Dynamical System and Excitation*

To illustrate the application of Melnikov-based open-loop control we assume that our system is described by Eq. 1 in which $f(z)=x-x^3$ (i.e., Eq. 1 is the Duffing-Holmes equation), and that the spectral density of the forcing $G(t)$ is

$$2\pi\Psi(\omega)=\begin{cases} 0.03990\ell n(\omega)+0.12829 & 0.04\leq\omega\leq 0.4 \\ 0.05755\ell n(\omega)+0.14493 & 0.4\leq\omega\leq 1.2 \\ -0.38301[\ell n(\omega)]^2+1.06192\ell n(\omega)-0.02941 & 1.2\leq\omega\leq 15.4 \end{cases}$$

(Fig. 1). A rescaled version of this spectrum approximates low-frequency fluctuations of the horizontal wind speed [3]. For our system the Melnikov relative scale factor is $S(\omega) = (2)^{1/2}\pi\omega \mathrm{sech}(\pi\omega/2)$ (Fig. 2a), and K=4/3 [1]. The spectral density of the Melnikov process for $\gamma=1$ (see Eq. 3), is shown in Fig. 2b.

4.1.2 *Types of Open-loop Control*

One possible type of open-loop control force has the expression $-\epsilon\gamma_a G(t-t_o)$ *and* seeks to counteract the excitation by applying a control force proportional and of opposite sign to $\gamma G(t)$. We refer to this as type *(a)* control. The smaller the lag t_o, the more effective the control.

A more efficient open-loop control force is one in which the information provided by the Melnikov relative scale factor $S(\omega)$ is utilized as follows. Figures 1 and 2 show that, owing to their suppression by $S^2(\omega)$, spectral components with frequencies $0 \leq \omega \leq \omega_1$, where $\omega_1 < 0.3$, say, and frequencies $\omega > \omega_2$, where $\omega_2 = 2.5$, say, contribute little to the spectral density of the Melnikov process. We refer to these components as *ineffective*. Our objective is to increase the system's mean exit time τ_e, and as indicated earlier to accomplish this we must increase the mean zero upcrossing time of the system's Melnikov process, τ_u, that is, we must reduce the spectral density of the controlled system's Melnikov process. We can do so -- more effectively than by applying a control force of type *(a)* -- by passing the signal $-\gamma_b G(t-\tau_o)$ through an ideal filter that suppresses the ineffective components. We refer to this control force as type *(b)*.

We now consider a third type of control force, obtained by passing the signal $-\epsilon\gamma_c G(t-t_o)$ through a realistic, practical filter, the impulse response of which is shown in Fig. 3 (a=0.1, b=2.25). This control force is referred to as type *(c)*.

The type *(c)* control force can be improved upon by passing the signal $-\epsilon\gamma_d G(t-t_o)$ through the filter of case *(c)*, and then suppressing from the output all ineffective Fourier components while leaving the other components unchanged. We refer to this force as type *(d)*.

4.1.3 Simulation Results

In all the simulations we assumed $t_o=0.1$, and $\gamma_b=0.5$, $\gamma_d=0.5$, $\epsilon=0.1$, $\beta=0.45$. Control force *(a)* was chosen so that it has the same average power as force *(b)*; this yielded $\gamma_a=0.195$. A similar criterion applied to the forces *(c)*, *(d)* resulted in $\gamma_c=0.167$. (To within a constant the average power is the variance of the control force.) Simulation results are summarized in Fig. 4. For example, Fig. 4 shows that, given the external excitation $\sigma \equiv \epsilon\gamma = 0.15$, the escape rate reduction due to the use of a control force type *(b)* is about 20 times larger than that due to a control force type *(a)* having the same average power; and control force type *(d)* is almost five times more effective than control force type *(c)* with the same power. Note that the effectiveness of the control force increases as σ decreases.

The simulation results show that the information inherent in the Melnikov scale factors can be utilized to obtain relatively effective open-loop control systems aimed at reducing escape rates. The extent to which this is the case depends upon the spectral density of the excitation, the system characteristics as reflected by the Melnikov relative scale factor, the lag time t_o, and the properties of the filter being used.

5. Snap-through of Buckled Column with Continuous Mass Distribution, Excited by Distributed Stochastic Load

We now illustrate the application of the Melnikov approach to a spatially-extended stochastic dynamical system. We obtain a stochastic counterpart of the Melnikov necessary condition for chaos --and snap-through-- derived by Holmes and Marsden

(1981) [10] for the harmonic loading case. As in Section 3, our approach can yield a lower bound for the probability that snap-through cannot occur during a specified time interval. For excitations with finite-tailed marginal distribution, a simple criterion can be obtained that guarantees the non-occurrence of snap-through.

Assume that: (a) the mechanical properties of the column are uniform over its length, (b) the material is linearly elastic, (c) following the initial, static deformation of the column due to buckling the distance between the column supports is fixed, and (d) the column deformations are sufficiently small that, in the Taylor expansion of the projection of the elemental deformed column length on the line joining the column supports, terms of power higher than two can be neglected. The equation of motion of the column is then [10, 14]

$$z_{tt} + z_{yyyy} + \{\Gamma - \xi \int_o^1 z_y^2(\zeta,t)d\zeta\}z_{yy} = \epsilon\{R(y,t) - \beta z_t\} \tag{4a}$$

$$R(y,t) = \gamma(y)cos(\omega_o t) + \rho(y)G(t) \tag{4b}$$

where $z(y,t)=Z(Y,\tau)/\Delta$, Z=deflection at time τ, Y=coordinate along column length ℓ, $y=Y/\ell$, $\Delta=Z_o(\ell/2)$ is the static deflection of the column $Z_o(Y)$ at midlength, t and τ=dimensionless and dimensional time, respectively, $\Gamma=P_o\ell^2/EI$, E=Young's modulus, I=moment of inertia of column cross-section with respect to weak axis,

$$P_o=P_{cr}+[EA/2\ell]\int_o^{\ell} (dZ_o/dY)^2dY, \tag{4c}$$

$P_{cr}=k\pi^2EI/\ell^2$ is Euler's critical buckling load, k=coefficient dependent upon the boundary conditions (for columns hinged at both ends $k=1$), A=cross-sectional area, $\xi=\frac{1}{2}\Delta^2A/I$, $\epsilon\beta=c\ell^2/[mEI]^{1/2}$, c=viscous damping coefficient, m=column mass per unit length, $t=\omega_1\tau$ is the nondimensional time, $\omega_1^2=(EI/\ell^4m)$, $\epsilon\gamma(y) = f(Y)\ell^4m/(EI\Delta)$, $f(Y)$=amplitude of harmonic force per unit length, $G(t)$=nondimensional nonperiodic function, $\epsilon\rho(y)=s(Y)\ell^4m/(EI\Delta)$, $s(Y)$=measure of nonperiodic force per unit length. Both ends of the column are assumed to be hinged, i.e., the boundary conditions are $z(0,t) = z(1,t) = z_{yy}(0,t) = z_{yy}(1,t) = 0$. The initial deflection $Z(Y,0)=Z_o(Y)$. For our boundary conditions $Z_o(Y)=\Delta sin(\pi Y/\ell)$. It can be easily verified that $\Gamma = \pi^2+\pi^2\xi/2$.

The eigenvalues of the linearized, unforced equation are [10]

$$\lambda_j=\pm\pi j(\Gamma-\pi^2j^2)^{1/2},\ j=1,2,... \tag{5a}$$

From the expression of Γ given earlier it follows that $\Gamma\geq\pi^2$. Since we assume the deflections are small, $\pi^2<\Gamma<4\pi^2$. Therefore the solution $z=0$ has one positive and one negative eigenvalue and the system with $\epsilon=0$ and $\xi>0$ has two nontrivial buckled equilibrium states. The system also has pure imaginary eigenvalues

$$\lambda_n = \pm \pi n(\Gamma - \pi^2 n^2)^{1/2}, \quad n = 2,3,... \tag{5b}$$

The expansion of $z(y,t)$ in the eigenfunctions of the linearized problem

$$z(y,t) = \sum_{j=1}^{\infty} a_j(t) \sin(j\pi y),$$

used with the Galerkin method, yields

$$\ddot{a}_j + \epsilon\beta\dot{a}_j + (j\pi)^2\{(j\pi)^2 - [\Gamma - (\xi\pi^2/2)\sum_{k=1,2,..} k^2 a_k^2]\}a_j = \epsilon[2\phi_j \cos(\omega_o t) + \psi_j G(t)] \tag{6}$$

where $\phi_j = \int_0^1 \gamma(y)\sin(j\pi y)dy$, $\psi_j = \int_0^1 \rho(y)\sin(j\pi y)dy$.

The unperturbed counterpart of Eq. 4a has homoclinic orbit coordinates [11]

$$z_o(y,t) = (2)^{1/2} \sin(\pi y) \operatorname{sech}[t\pi(\Gamma - \pi^2)^{1/2}]$$

$$\dot{z}_o(y,t) = -(2)^{1/2}\pi(\Gamma - \pi^2)^{1/2} \sin(\pi y) \operatorname{sech}[t\pi(\Gamma - \pi^2)^{1/2}] \tanh[t\pi(\Gamma - \pi^2)^{1/2}].$$

The Melnikov function for the harmonically excited system can be written as

$$M(t) = \int_{-\infty}^{\infty}\int_0^1 [R(y,\theta)\dot{z}_o(y,\theta - t) - \beta\dot{z}_o^2(y,\theta - t)]dy\,d\theta \tag{7}$$

where $R(y,t)$ is given by Eq. 4b.

We now consider Eq. 6 and let $\rho(y)=0$. If the non-resonance condition $\omega_o^2 \neq -\lambda_j^2$ holds, Eq. 6 has unique solutions of $O(\epsilon)$; otherwise the linearized counterparts of Eqs. 6 would have solutions of $O(1)$. This would violate a basic assumption of Melnikov theory [10]. If $\rho(y) \neq 0$, for excitations $G(t)$ with continuous spectral density it can be shown that the solutions of the linearized counterparts of the Galerkin equations are of $O(\epsilon)^{1/2}$ [15]. For sufficiently small ϵ those solutions will be as small as desired, and non-resonance conditions associated with $G(t)$ are not required for the assumptions of Melnikov theory to be satisfied.

For the particular case of dichotomous coin-toss square wave noise, following steps similar to those of [5], it can be shown that Eq. 7 yields the following criterion guaranteeing the non-occurrence of snap-through

$$\rho_o \leq 2.584 \xi^{1/2} \beta \tag{8}$$

The validity of this criterion was verified by numerical simulations via the system's Galerkin equations. For additional details, see [6,16].

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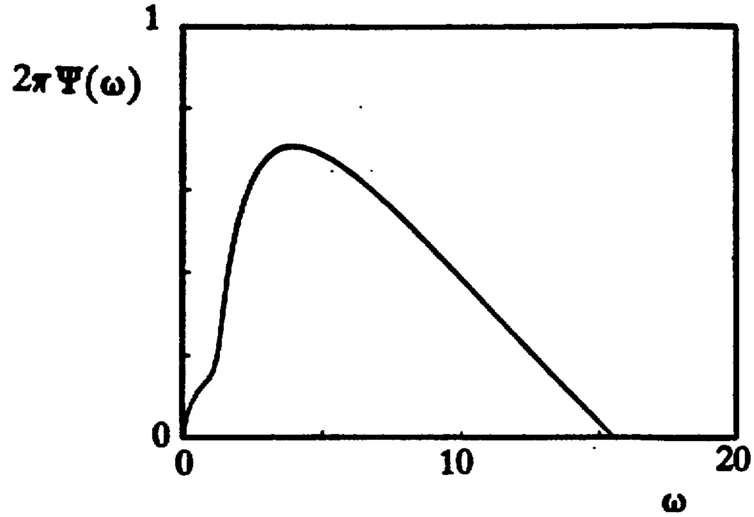

Fig. 1. Spectral density of uncontrolled system's excitation G(t)

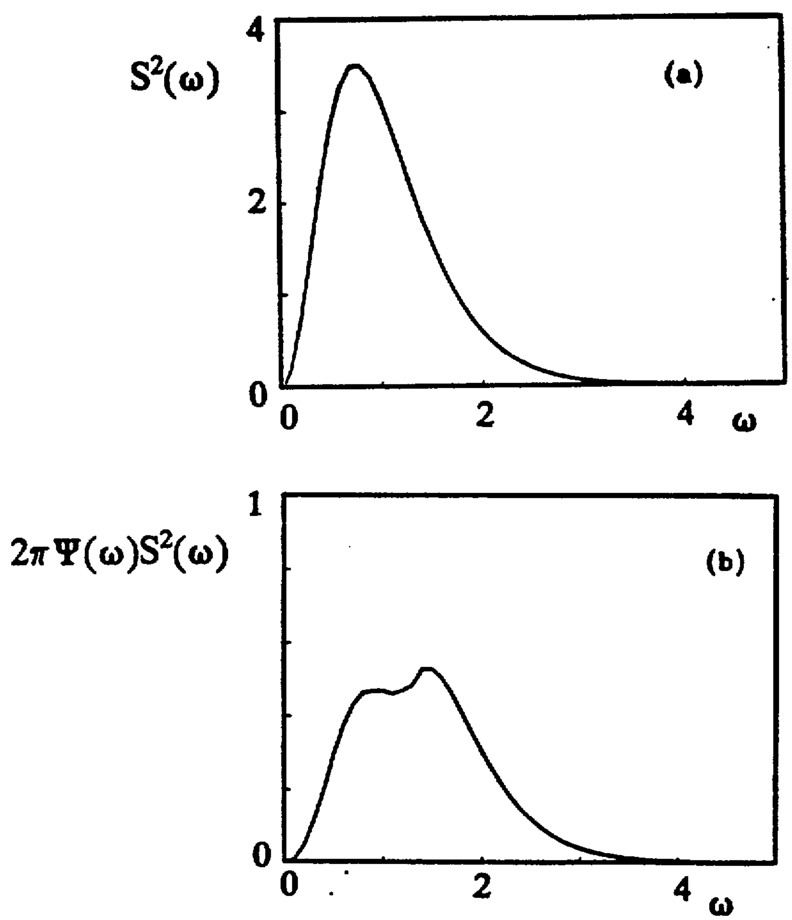

Fig. 2. (a) Melnikov relative scale factor;
(b) spectral density of Melnikov process.

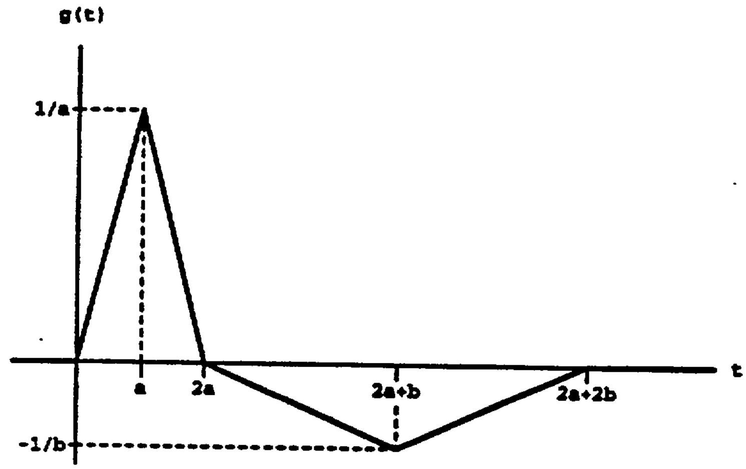

Fig. 3. Impulse response function of filter with initial response and recoil

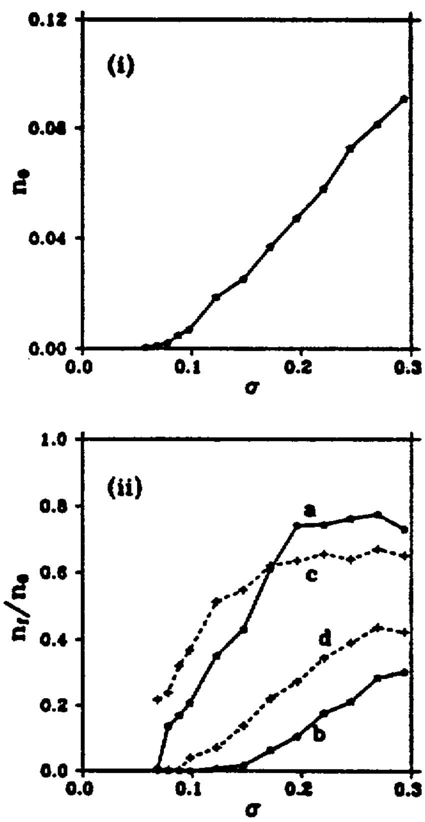

Fig. 4 (a) Escape rate n_o for uncontrolled oscillator; (b) ratio n_f/n_o between escape rates for controlled and uncontrolled oscillator.