Melnikov Process for Stochastically Perturbed, Slowly Varying Oscillators: Application to a Model of Wind-Driven Coastal Currents

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The stochastic Melnikov approach is extended to a class of slowly varying dynamical systems. It is found that (1) necessary conditions for chaos induced by stochastic perturbations depend on the excitation spectrum and the transfer function in the expression for the Melnikov transform; (2) the Melnikov approach allows the estimation of lower bounds for (a) the mean time of exit from preferred regions of phase space, and (b) the probability that exits from those regions cannot occur during a specified time interval. For a system modeling wind-induced currents, the deterministic Melnikov approach would indicate that chaotic transport cannot occur for certain parameter ranges. However, the more realistic stochastic Melnikov approach shows that, for those same parameter ranges, the necessary conditions for exits during a specified time interval are satisfied with probabilities that increase as the time interval increases.

Introduction

The Melnikov approach is a technique providing necessary conditions for the occurrence of chaos in a class of dynamical systems. Until recently it was considered to be applicable only to deterministic systems: deterministic chaos and stochastic motions were viewed as distinct and were analyzed from different, indeed contrasting points of view.

Beigie et al. (1991) extended Melnikov theory for a class of single-degree-of-freedom systems from the case of periodic to the case of quasi-periodic excitation. Their work allowed a further extension to the case of stochastic excitation (Frey and Simiu, 1993; Simiu and Frey, 1996). This extension used the representation of Gaussian excitations by Shinozuka processes (Shinozuka, 1971; Shinozuka and Deodatis, 1991), which have two properties needed for the application of Melnikov theory: uniform continuity and uniform boundedness. One result of this extension is that, under certain conditions, a motion can be both stochastic (i.e., induced by a realization of a stochastic process) *and* chaotic (i.e., sensitive to initial conditions).

Necessary conditions for chaos indicate the range of system parameters for which exits from preferred regions of phase space cannot occur. The Melnikov approach can thus help to study the exit problem for certain types of nonlinear stochastic systems to which no other analytical tools are applicable, e.g., multistable systems with dichotomous noise (Simiu and Hagwood, 1994). Examples of applications of the Melnikov approach include the rocking response of rigid objects to earthquakes (Yim and Lin, 1992) and the prediction of vessel capsizing in random beam seas (Hsieh, Troesch, and Shaw, 1994).

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Necessary conditions for the occurrence of chaos in a class of slowly varying oscillators were obtained by Wiggins and Holmes (1987) and Wiggins and Shaw (1988) for the case of small periodic perturbations. This paper extends these authors' theory for perturbations that, rather than being periodic, are quasi-periodic or stochastic. Our results are based on arguments similar to those used by Beigie et al. (1991) and Frey and Simiu (1993) for single-degree-of-freedom systems, and on the application of the averaging theorem to systems with quasiperiodic perturbations (Verhulst, 1990).

We then consider a model of coastal currents over topography. We adopt the hydrodynamical model studied by Allen et al. (1991) for the ideal case of harmonic wind forcing, and apply our results to the more realistic case of forcing by randomly fluctuating wind.

Dynamical Systems

The systems we study are of the form

$$\dot{x} = \frac{\partial}{\partial y} H(x, y, z) + \epsilon g_1(x, y, z, t; \boldsymbol{\mu})$$

$$\dot{y} = -\frac{\partial}{\partial x} H(x, y, z) + \epsilon g_2(x, y, z, t; \boldsymbol{\mu})$$

$$\dot{z} = \epsilon g_3(x, y, z, t; \boldsymbol{\mu}) \tag{1.1}$$

where ϵ is small, the right-hand side is C^r differentiable ($r \geq 2$), $H(x, y, z)$ is a Hamiltonian with parameter z, $\boldsymbol{\mu}$ is a vector of parameters, and g_i ($i = 1, 2, 3$) are the perturbation functions. It is assumed that there exists an open interval $J \subset R$ such that, for every $z \in J$, the unperturbed system possesses a homoclinic orbit to a hyperbolic saddle point. In the full three-dimensional phase space the unperturbed system has a normally hyperbolic invariant one-dimensional manifold, $\gamma(z)$, assumed to be connected, and given by the union of saddle points of the one-parameter family of planar systems. $\gamma(z)$ has two-dimensional stable and unstable manifolds (denoted by $W^s(\gamma)$, $W^u(\gamma)$, respectively), such that their intersection $W^s(\gamma) \cap W^u(\gamma) \equiv \Gamma$ is the union of the homoclinic orbits of the planar systems.

Functions g_i ($i = 1, 2, 3$) With Common Period T

In this section we review pertinent material from Wiggins and Holmes (1987) and Wiggins and Shaw (1988). Let T^1 be the circle of unit length, and write Eq. (1) in the form

$$\dot{x} = \frac{\partial}{\partial y} H(x, y, z) + \epsilon g_1(x, y, z, \theta; \boldsymbol{\mu})$$

$$\dot{y} = -\frac{\partial}{\partial x} H(x, y, z) + \epsilon g_2(x, y, z, \theta; \boldsymbol{\mu})$$

$$\dot{z} = \epsilon g_3(x, y, z, \theta; \boldsymbol{\mu})$$

$$\dot{\theta} = \omega \tag{2.1}$$

where $\omega = 2\pi/T$. The normally invariant set at $\epsilon = 0$ is denoted by $M = (\gamma, \theta) = \gamma x T^1$, or

$$M = \left\{ (\gamma(z), \theta) | \gamma(z) = (x(z), y(z), z), \frac{\partial}{\partial y} H(x, y, z) = -\frac{\partial}{\partial z} H(x, y, z) = 0, \left.\frac{\partial(\partial H/\partial y, -\partial H/\partial x)}{\partial(x, y)}\right|_{\gamma(z)} < 0, \theta \in T^1, z \in J \right\}. \tag{2.2}$$

Persistence Theorem. There exists ϵ_0 such that for $0 < \epsilon < \epsilon_0 \leq 1$ there exists a normally invariant manifold

$$M_\epsilon = \{(\gamma(z, \theta; \epsilon), \theta) = (\gamma(z) + O(\epsilon), \theta) | \theta \in T^1, z \in J\} \tag{2.3}$$

where $\gamma(z, \theta; \epsilon)$ is a C^r function of z and ϵ. M_ϵ has local stable and unstable manifolds $W^s_{loc}(M_\epsilon)$, $W^u_{loc}(M_\epsilon)$, which are C^r-close to the local stable and unstable manifolds of M, denoted $W^s_{loc}(M)$, $W^u_{loc}(M)$, respectively.

Flow on M_ϵ. For the perturbed vector field restricted to M_ϵ, Eqs. (2.1) and (2.3) yield

$$\dot{z} = \epsilon g_3(\gamma(z), \theta) + O(\epsilon^2). \tag{2.4}$$

($\boldsymbol{\mu}$ is omitted for simplicity). Consider the averaged system

$$\dot{z} = \epsilon \overline{g_3[\gamma(z)]}, \quad \overline{g_3[\gamma(z)]} = \frac{1}{T}\int_0^T g_3[\gamma(z), \theta] d\theta. \tag{2.5a, b}$$

If there exists $z_0 \in J$ such that

$$\overline{g_3(\gamma(z_0))} = 0, \quad d\overline{[g_3(\gamma(z_0))]}/dz \neq 0, \tag{2.6a, b}$$

then $(\gamma(z_0, \theta; \epsilon), \theta) = (\gamma(z_0) + O(\epsilon), \theta)$ is a hyperbolic periodic orbit on M_ϵ with period T. The orbit restricted to M_ϵ is stable or unstable for, respectively,

$$d\overline{[g_3(\gamma(z_0))]}/dz < 0 \quad \text{or} \quad d\overline{[g_3(\gamma(z_0))]}/dz > 0. \tag{2.7a, b}$$

Distance Between Stable and Unstable Manifolds and Necessary Condition for Chaos. We define the cross section through the unperturbed vector field $\Sigma = \{(x, y, z, \theta) | \theta = \theta_0 \in (0, 1)\}$ and the Poincaré map of Σ into itself, $P: \Sigma \to \Sigma$. If there exists $z_0 \in J$ satisfying Eqs. (2.6), the Poincaré section through the hyperbolic periodic orbit on M_ϵ is a hyperbolic fixed point $\gamma(z_0, \theta_0; \epsilon)$ having: a two-dimensional stable manifold and a one-dimensional unstable manifold if Eq. (2.7*a*) holds; or a one-dimensional stable manifold and a two-dimensional unstable manifold if Eq. (2.7*b*) holds. To first order, the distance between the stable and unstable manifolds of the perturbed system is proportional to the Melnikov function

$$m(s, \theta_0) = \int_0^\infty (\nabla H \cdot \mathbf{g})(q_0^{z_0}(t), \omega t + (\omega s + \theta_0)) dt - \frac{\partial H}{\partial z}(\gamma(z_0)) \int_0^\infty g_3(q_0^{z_0}(t), \omega t + (\omega s + \theta_0)) dt \tag{2.8}$$

where $\nabla H = (\partial H/\partial x, \partial H/\partial y, \partial H/\partial z)$, $\mathbf{g} = (g_1, g_2, g_3)$, z_0 satisfies Eqs. (2.6), $q_0^{z_0}(t)$ is the homoclinic orbit in the unperturbed system connecting the saddle point $\gamma(z_0)$ to itself, and (s, θ_0) specify a point on M and are therefore constant in the integral expression (Wiggins and Holmes, 1987). The Melnikov function can be interpreted as the sum of the outputs of three linear filters with inputs g_1, g_2, g_3, respectively.

If there exists an $\underline{s}$ such that, for fixed θ_0,

$$m(\underline{s}, \theta_0) = 0 \quad \text{and} \quad (\partial/\partial s)[m(\underline{s}, \theta_0)] \neq 0 \tag{2.9a, b}$$

then for sufficiently small ϵ near this point the stable and unstable manifolds of the fixed point $\gamma(z_0, \theta_0; \epsilon)$ intersect transversely. This implies the existence of transverse homoclinic points. The dynamics of the perturbed system can then give rise to a three-dimensional mapping with Smale horseshoes. This mapping has the same dynamics as a shift map acting on the space of bi-infinite sequences of 0's and 1's. The fact that the shift map has chaotic dynamics establishes that Eqs. (2.9) are a necessary condition for the perturbed system to be chaotic. This result is a special case of the Smale-Birkhoff theorem.

Quasi-Periodic Functions g_i ($i = 1, 2, 3$)

We now consider Eqs. (1.1), and assume that the perturbations are quasi-periodic in t, that is,

$$g_i(x, y, z, t) = g_{i0}(x, y, z) + \sum_{j=l_{(i-1)}+1}^{l_i} g_{ij}(x, y, z) \cos(\omega_j t + \theta_{j0})$$

$$(i = 1, 2, 3; l_0 = 0 < l_1 < l_2 < l_3 \equiv l \geq 2) \tag{3.1}$$

$(\omega_j t + \theta_{j0} \equiv \theta_j)$, where in general $\omega_1, \omega_2, \ldots, \omega_n$ are incommensurate. To study this system we follow the approach used by Beigie et al. (1991) for second-order systems with quasi-periodic vector fields. For Eq. (1.1) with quasi-periodic functions g_i the flow is no longer characterized by a Poincaré map, as is the case for periodic vector fields, but by a bi-infinite sequence of nonautonomous maps, where the maps are functions of time that change at each iteration.

Associated with the nonautonomous system involving l frequencies is the $(l + 3)$-dimensional autonomous system

$$\dot{x} = \frac{\partial}{\partial y} H(x, y, z) + \epsilon g_1(x, y, z, \theta_1, \theta_2, \ldots, \theta_{l1}; \boldsymbol{\mu})$$

$$\dot{y} = -\frac{\partial}{\partial x} H(x, y, z) + \epsilon g_2(x, y, z, \theta_{(l1+1)}, \theta_{(l1+2)}, \ldots, \theta_{l2}; \boldsymbol{\mu})$$

$$\dot{z} = \epsilon g_3(x, y, z, \theta_{(l2+1)}, \theta_{(l2+2)}, \ldots, \theta_l; \boldsymbol{\mu})$$

$$\dot{\theta}_1 = \omega_1$$

$$\cdot \tag{3.2}$$

$$\cdot$$

$$\dot{\theta}_l = \omega_l.$$

The autonomous system's phase space for Eqs. (3.2) is $R^3 \times T^l$, where T^l is the l-torus. The expression

$$(x(t), y(t), z(t), \theta_1(t), \ldots, \theta_l(t)) = (\phi(t, t_0 = 0, x(0), y(0), z(0)), \omega_1 t + \theta_{10}, \ldots, \omega_l t + \theta_{l0}) \tag{3.3}$$

solves Eq. (3.2) and is referred to as the trajectory of Eq. (3.2) through $(x(0), y(0), z(0), \theta_{10}, \ldots, \theta_{l0})$ at time $t = 0$. For the unperturbed counterpart of Eqs. (3.2) the autonomous phase space contains for every $z \in J$ an l-dimensional normally hyperbolic invariant set $M^l = (\gamma, \theta_1, \ldots, \theta_l) = \gamma x T^l$, or

$$M^l = \{(x, y, z, \theta_1, \ldots, \theta_l) | (x, y, z) = \gamma(z), z \in J\} \quad (3.4)$$

whose $(l + 1)$-dimensional stable and unstable manifolds, denoted $W^s(M^l)$ and $W^u(M^l)$, coincide along the $(l + 1)$-dimensional manifold given by

$$W^s(M^l) \cap W^u(M^l)$$
$$= \{(x, y, z, \theta_1, \ldots, \theta_l) | (x, y, z) = q_0^{\bar{z}}(t), t \in R\}. \quad (3.5)$$

Persistence Theorem. There exists ϵ_0 such that, for $0 < \epsilon < \epsilon_0 \leq 1$, there exists a normally invariant manifold

$$M_\epsilon^l = \{(\gamma(z, \theta_1, \theta_2, \ldots, \theta_l; \epsilon), \theta_1, \ldots, \theta_l)$$
$$= (\gamma(z) + O(\epsilon), \theta_1, \ldots, \theta_l) |$$
$$\times \theta_1, \ldots, \theta_l \in T^l, z \in J\} \quad (3.6)$$

where $\gamma(z, \theta_1, \theta_2, \ldots, \theta_l; \epsilon)$ is a C^r $(r \geq 2)$ function of z and ϵ. M_ϵ^l has local stable and unstable manifolds $W_{loc}^s(M_\epsilon^l)$ and $W_{loc}^u(M_\epsilon^l)$, which are C^r-close to the local stable and unstable manifolds of M^l, $W_{loc}^s(M^l)$ and $W_{loc}^u(M^l)$, respectively. The proof of the theorem is independent of whether g_i are periodic or quasi-periodic; it is the same as in Wiggins and Holmes (1987).

Flow on M_ϵ^l. For the perturbed vector field restricted to M_ϵ^l there follows from Eqs. (3.2) and (3.6)

$$\dot{z} = \epsilon g_3(\gamma(z), \theta_1, \ldots, \theta_l) + O(\epsilon^2). \quad (3.7)$$

We now consider the averaged system

$$\dot{z} = \epsilon \overline{g_3[\gamma(z)]}, \quad (3.8a)$$

$$\overline{g_3[\gamma(z)]} = \sum_{j=l_2+1}^{l} \frac{1}{T_j} \int_0^{T_j} g_3(\gamma(z), \theta_1, \ldots, \theta_l) d\theta_j \quad (3.8b)$$

$(T_j = 2\pi/\omega_j)$. If there exists $z_0 \in J$ such that

$$\overline{g_3(\gamma(z_0))} = 0, \quad d[\overline{g_3(\gamma(z_0))}]/dz \neq 0 \quad (3.9a, b)$$

then $(\gamma(z_0, \theta_1, \ldots, \theta_l; \epsilon), \theta_1, \ldots, \theta_l) = (\gamma(z_0) + O(\epsilon), \theta_1, \ldots, \theta_l)$ is a hyperbolic T^l-torus on M_ϵ^l, and the orbit restricted to M_ϵ^l is stable or unstable for, respectively,

$$d[\overline{g_3(\gamma(z_0))}]/dz < 0 \text{ or } d[\overline{g_3(\gamma(z_0))}]/dz > 0. \quad (3.10a, b)$$

The proof of this proposition follows exactly the same steps as its counterpart for periodic functions g_i, with one exception: instead of the classical averaging theorem, in which averaging is defined as in Eq. (2.5b), the proof uses the averaging theorem as applied to quasiperiodic perturbations, where averaging is defined as in Eq. (3.8a)—see, e.g. (Verhulst, 1990, p. 154).

Distance Between Stable and Unstable Manifolds and Necessary Condition for Chaos. A global cross section of the autonomous phase space is defined by

$$\Sigma^{\theta j0}\{(x, y, z, \theta_1, \ldots, \theta_l) | \theta_j = \theta_{j0}\} \quad 1 \leq j \leq l. \quad (3.11)$$

The associated Poincaré map is

$$P_\epsilon: \Sigma^{\theta j0} \to \Sigma^{\theta j0} \quad (3.12)$$

$$(x, y, z, \boldsymbol{\theta}_0 + 2\pi n \boldsymbol{\omega}/\omega_j) \to \{\phi[(2\pi/\omega_j)(n + 1),$$
$$(2\pi/\omega_j)n, x, y, z)], \boldsymbol{\theta}_0 + 2\pi n \boldsymbol{\omega}/\omega_j\} \quad (3.13)$$

where $\boldsymbol{\theta}_0 \equiv (\theta_{10}, \ldots, \theta_{(j-1)0}, \theta_{(j+1)0}, \ldots, \theta_{l0})$, $\boldsymbol{\omega} \equiv (\omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_l)$. Since this map is obtained from an autonomous system, it is independent of n. The Poincaré map is equivalent to sampling the trajectories of the system (3.2) at time intervals $2\pi/\omega_j$. For each z the Poincaré map of the unperturbed system has a normally hyperbolic $(l - 1)$-torus

$$\tau_0 = T_0 \cap \Sigma^{\theta j0} = \{(x, y, z, \theta_1, \ldots, \theta_l) | (x, y, z) = \gamma(z),$$
$$\theta_j = \theta_{j0}\} \quad (3.14)$$

with an l-dimensional homoclinic manifold

$$W^s(\tau_0) \cap W^u(\tau_0) = \{(x, y, z, \theta_1, \ldots, \theta_l) | (x, y, z)$$
$$= q_0^{\bar{z}}(t), \theta_j = \theta_{j0}, t \in R\}.$$

We now consider the generalized Melnikov function

$$m(s, \theta_1, \theta_2, \ldots, \theta_l)$$
$$= \int_0^\infty (\nabla H \cdot \mathbf{g})(q_0^{z0}(t), \omega_1 t + (\omega_1 s + \theta_1),$$
$$\omega_2 t + (\omega_2 s + \theta_2), \ldots, \omega_l t + (\omega_l s + \theta_l)) dt$$
$$- \frac{\partial H}{\partial z}(\gamma(z_0)) \int_0^\infty g_3(q_0^{z0}(t)), \omega_1 t + (\omega_1 s + \theta_1),$$
$$\omega_2 t + (\omega_2 s + \theta_2), \ldots, \omega_l t + (\omega_l s + \theta_l)) dt \quad (3.15)$$

where $\nabla H = (\partial H/\partial x, \partial H/\partial y, \partial H/\partial z)$, $\mathbf{g} = (g_1, g_2, g_3)$, z_0 satisfies Eqs. (3.9), $q_0^{z0}(t)$ is the homoclinic orbit in the unperturbed system connecting $\gamma(z_0)$ to itself, and $(s, \theta_1, \theta_2, \ldots, \theta_{(l-1)}, \theta_l)$ specify a point on the manifold $W^s(M^l) \cap W^u(M^l)$ of the unperturbed system and are therefore constant in the integral expression. In the Poincaré section, the generalized Melnikov function has the same expression as in Eq. (3.15), except that θ_j takes on the fixed value θ_{j0}. Like the Melnikov function for periodic g_i's, the generalized Melnikov function may be interpreted as the sum of the outputs of three linear filters with inputs g_1, g_2, g_3, respectively.

For $\boldsymbol{\theta} \equiv (\theta_1, \theta_2, \ldots, \theta_{(j-1)}, \theta_{(j+1)}, \ldots, \theta_l) \in T^{l-1}$, we refer to the plane $\chi(x, y, z; \underline{\boldsymbol{\theta}}) \equiv \{x, y, z, \boldsymbol{\theta} | \boldsymbol{\theta} = \underline{\boldsymbol{\theta}}\}$ as a three-dimensional phase slice of the Poincaré section $\Sigma^{\theta j0}$. If there exists a point $(\underline{s}, \underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_{(j-1)}, \underline{\theta}_{(j+1)}, \ldots, \underline{\theta}_l)$ such that

$$m(\underline{s}, \underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_{(j-1)}, \theta_{j0}, \underline{\theta}_{(j+1)}, \ldots, \underline{\theta}_l) = 0, \quad \text{and} \quad (3.16a)$$

$$(\partial/\partial s)[m(\underline{s}, \underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_{(j-1)},$$
$$\theta_{j0}, \underline{\theta}_{(j+1)}, \ldots, \underline{\theta}_l)] \neq 0 \quad (3.16b)$$

then, for sufficiently small ϵ, near this point the generalized Melnikov function in the Poincaré has simple zeros, that is, the stable and unstable manifolds of the Poincaré section of M_ϵ^l intersect transversely, i.e., there exist transverse homoclinic points. A phase slice in the quasiperiodic perturbation case differs from a Poincaré map for the periodic case in that the phase slice changes at each successive time $2\pi/\omega_{j0}$, whereas the Poincaré map repeats itself at successive times $2\pi/\omega$. This observation allows the extension of the Smale-Birkhoff theorem from the case of periodic perturbations to the case of quasi-periodic perturbations (Beigie et al., 1991). As is the case for the periodically perturbed system studied by Wiggins and Shaw (1988), the dynamics of the quasi-periodically perturbed system having a Melnikov function with simple zeros can give rise to a mapping with Smale horseshoes which is equivalent to a shift map acting on the space of bi-infinite sequences of 0's and 1's. This establishes that Eqs. (3.15) are a necessary condition for the system with quasi-periodic perturbation to be chaotic.

Necessary conditions for the occurrence of chaos follow immediately from Eqs. (3.15) for (a) perturbation functions g_1, g_2, g_3 with incommensurate periods T_1, T_2, T_3, respectively; (b) one quasi-periodic function and two periodic functions having different or equal periods; (c) two quasi-periodic functions and one periodic function; (d) any set of perturbation functions that may be approximated by finite sums of harmonic terms.

Stochastic Perturbations

We now assume g_i $(i = 1, 2, 3)$ are additive Gaussian processes with specified spectral densities. Such processes may be modeled effectively by ensembles of harmonic sums. Since the proof of the persistence theorem—a prerequisite for Melnikov analysis—requires that the perturbations be bounded and uniformly continuous, we use the Shinozuka (1971) representation of Gaussian noise, which has the requisite properties

$$G_i(t) = (1/2\pi)^{1/2}(2/l)^{1/2} \sum_{n=1}^{l} \cos(\omega_{in}t + \theta_{in0}) \quad (i = 1, 2, 3) \quad (4.1)$$

where l is a parameter of the model (the approximation of a Gaussian process by the model improves as l increases), $\{\omega_{in}, \theta_{in0};\ n = 1, \ldots, l\}$ are independent random variables, $\{\theta_{in0},\ n = 1, \ldots, l\}$ are identically uniformly distributed over the interval $[0, 2\pi]$, $\{\omega_{in};\ n = 1, \ldots, l\}$ are non-negative with common distribution equal to the spectral density of the process, Ψ_{i0}, and

$$\frac{1}{2\pi}\int_0^\infty \Psi_{i0}(\omega)d\omega = 1. \quad (4.2)$$

In this model all the harmonics have equal amplitude $(1/\pi l)^{1/2}$, and the number of harmonics with frequencies contained in a given interval is proportional to the area under the spectral curve within that interval. Equation (4.1) can approximate a Gaussian process as closely as desired provided that the finite value of l is sufficiently large. We assume

$$g_i(x, y, z, t) = g_{i0}(x, y, z) + \sigma_i G_i(t) \quad (4.3)$$

$(i = 1, 2, 3)$. Each realization in Eq. (4.3) has a different set of random parameters $\{\omega_{in}, \theta_{in0};\ n = 1, \ldots, l\}$. Since l is finite, the arguments presented in the previous section for quasiperiodic excitations apply without change for stochastic excitations approximated by the Shinozuka model as in Eq. (4.2) (Frey and Simiu, 1993). To each path there corresponds a Melnikov function similar to Eq. (3.15). We noted earlier that the Melnikov function is the sum of the outputs of three linear filters with inputs g_1, g_2, g_3. For sufficiently large l, each of the three sums approaches a Gaussian process, the mean and variance of which can easily be calculated since the filter properties are known. Therefore in the limit of large l the Melnikov process is Gaussian, and the probability that it has simple zeros is one—no matter how small the noise—provided that the time interval being considered is infinitely long (Frey and Simiu, 1993). However, the probability that a path of the Melnikov process has simple zeros during a finite time interval T is $p_T < 1$. Let the mean and standard deviation of the Melnikov process be $E[m]$ and σ_m. If the ratio $k = E[m]/\sigma_m$ is sufficiently large (say, $k > 2$), then p_T can be closely approximated by using: (a) the Kac-Rice formula (Rice, 1958) for the rate of upcrossing, $E(k)$, of a threshold k by a standardized Gaussian process with one-sided spectral density $\Psi_m(\omega)$, and (b) the assumption that the upcrossing is a rare event described by a Poisson process. Then

$$p_T = 1 - \exp[-E(k)T], \quad E(k) = \nu \exp(-k^2/2) \quad (4.4a, b)$$

$$\nu = (1/2\pi)\left\{\left[\int_0^\infty \omega^2\Psi_m(\omega)d\omega\right] \Big/ \left[\int_0^\infty \Psi_m(\omega)d\omega\right]\right\}^{1/2}. \quad (4.5)$$

Let t_{ex} be the mean time spent by the system in a region of phase space associated with a potential well before it exits from that region. In the limit of weak perturbations t_{ex} must be at least as large as $1/\nu$ (there can be no exit from that region as long as the stable and unstable manifolds do not intersect). Therefore $1/\nu$ is a lower bound for t_{ex}, and p_T is a lower bound for the probability that $t_{ex} > T$ (Simiu and Frey, 1996).

The case of multiplicative noise involves a simple modification of the Melnikov process filter (Frey and Simiu, 1995).

Necessary Conditions for Chaos in a Model of Wind-Induced Coastal Current

Allen et al. (1991) studied a model of ocean flow over a continental margin with variable bottom topography under wind forcing fluctuating harmonically. Our extension of the results by Wiggins and Holmes (1987) and Wiggins and Shaw (1988) to the case of stochastic forcing allows us to examine the model under the assumption that the wind fluctuations are random.

Fluctuating Wind and Surface Wind Stresses. We consider the horizontal wind speed fluctuations spectrum developed by Van der Hoven (1957). The spectrum has three main parts: one with a peak at a period of about four days, a second, known as the spectral gap, having negligible energy and extending over periods of about five hours to three minutes, and a third, with a peak at a period of about one minute, where fluctuations have relatively small spatial coherence and therefore a negligible overall effect at the scale of our problem (Simiu and Scanlan, 1986). The relevant part of the spectrum is, approximately,

$$S(\omega) = \begin{cases} 0.2823 \ln(\omega) + 1.300 & 0.01 \le \omega \le 0.10 \\ 0.4072 \ln(\omega) + 1.599 & 0.10 \le \omega \le 0.30 \\ -2.71[\ln(\omega)]^2 + 5 & 0.30 \le \omega \le 3.85 \end{cases} \quad (5.1)$$

where $\omega = \Omega/\Omega_{pk}$, Ω is the dimensional frequency and $\Omega_{pk} \approx 2\pi/(\text{four days})$ is the dimensional frequency corresponding to the spectral peak, which occurs at $\omega = 1$. The units of $S(\omega)$ are m^2/s^2 (Fig. 1). For the spectrum of Eq. (5.1) the standard deviation of the wind speed fluctuations is $\sigma_w \approx 1.33$ m/s.

Surface wind stresses are proportional to the square of the wind speeds (Simiu and Scanlan, 1986). We assume that the effect of thermal stratification and of deviations from the prevailing wind direction is small. Mean wind speeds of about 6 m/s are consistent with the data used to obtain the Van der Hoven spectrum (NOAA, 1977), so that the coefficient of variation of the wind speed fluctuations is about 0.20. To a first approximation, one may therefore neglect the square of the wind speed fluctuations in the expansion of $[U + u(t)]^2$, where U and $u(t)$ are the mean and fluctuating wind speed (Simiu and Scanlan, 1986; Vaicaitis and Simiu, 1977). The normalized

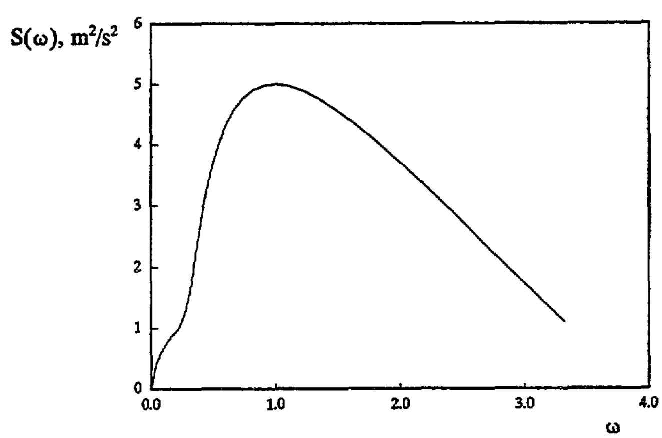

Fig. 1 Spectral density of wind speed fluctuations, $S(\omega)$

spectrum of the surface wind stresses is then $\Psi_0(\omega) \approx S(\omega)/\sigma_w^2$, and the wind stresses may be assumed to be Gaussian.

Offshore Flow Model. The nondimensional equations for the model flow are given by Eqs. (1.1), where x is a basic alongshore speed, y is proportional to the out-of-phase component of a stream function for motion due to the topography, and z is the energy-enstrophy:

$$g_1 = -rx + \tau_0 + \tau(t), \quad g_2 = -ry,$$

$$g_3 = -rz - \tfrac{1}{2}rx^2 + (x-1)[\tau_0 + \tau(t)] \tag{5.2}$$

$$H(x, y, z) = \tfrac{1}{2}y^2 + zx + \tfrac{1}{2}(\omega_0^2 - z)x^2 - \tfrac{1}{2}x^3 + (\tfrac{1}{8})x^4,$$

$$\omega_0^2 = 1 + \delta^2. \tag{5.3}$$

δ is the amplitude of the bottom topography corrugations, ϵr is a friction coefficient related to the eddy viscosity of the ocean flow, and $\epsilon\tau_0$ and $\epsilon\tau(t)$ are, respectively, the steady and fluctuating wind stress at the ocean surface (Allen et al., 1991).

We assume $\tau(t)$ is a Gaussian process with variance σ^2 and one-sided spectral density $\sigma^2\Psi_0(\omega)$, and approximate it as

$$\tau(t) = \sigma G(t) \tag{5.4}$$

where $G(t)$ has an expression similar to that of $G_i(t)$ in Eq. (4.12). The stochastic excitation is multiplicative (Eq. (5.2c)).

The dynamics of the unperturbed system was studied by Allen et al. (1991), who showed that for $z > z_c = (\tfrac{3}{2})\delta^{4/3} + \delta^2 - \tfrac{1}{2}$ the phase plane diagram x-y (z = const) has a saddle point and two elliptic centers with coordinates satisfying the expressions

$$x^3 - 3x^2 + 2(\omega_0^2 - z)x + 2z = 0, \quad y = 0, \tag{5.5a, b}$$

and contains an asymmetrical eight-shaped homoclinic orbit which asymptotically approaches the saddle point in forward and backward time. The two lobes of the homoclinic orbit separate three distinct regions of phase space, corresponding to three oscillatory regimes: one inside each lobe of the separatrix and a third outside the separatrix. The saddle point corresponds to the intermediate root of Eq. (5.5a). The saddle point and the separatrix depend continuously on z and form, respectively, a one-dimensional manifold $\gamma(z)$ of (x, y, z) points and a two-dimensional manifold of (x, y, z) points.

The existence of an oscillatory solution—a hyperbolic T^l-torus—among the solutions of the invariant manifold M_ϵ of the perturbed system is established by solving Eqs. (3.9). Since the integrals of the harmonic terms vanish, Eqs. (3.9) yield exactly the same solution as in the harmonic perturbation case, that is,

$$z = -\tfrac{1}{2}x^2 + (x-1)\tau_0/r, \tag{5.6}$$

$$d\overline{[g_3(\gamma(z_0))]}/dz < 0 \tag{5.7}$$

(Allen et al., 1991). The coordinates (x_0, y_0, z_0) that define to first order the hyperbolic T^l-torus on the invariant manifold M_ϵ^l are obtained from Eqs. (5.5) and (5.6), and from Eq. (5.7) it follows that the orbit restricted to M_ϵ^l is linearly stable.

For harmonic excitation $\tau(t) = \tau_1 \cos(\omega t)$, Allen et al. (1991) obtained for the Melnikov function the expression

$$m(s, \theta_0) = rC_1 + \tau_1 C_2(\omega)\cos(\omega s + \theta_0) \tag{5.8}$$

$$C_1 = C_1^\mp = (x_0 - \tau_0/r)[8d\tan^{-1}(V_{m\pm}/(2k_0)) - k_0 b] \tag{5.9}$$

$$C_2(\omega) = C_2^\mp(\omega) = -4\pi d\,\frac{\sinh[\omega\cos^{-1}(\alpha_\pm)/k_0]}{\sinh(\omega\pi/k_0)} \tag{5.10}$$

$$k_0 = (z_0 - \omega_0^2 + 3x_0 - (\tfrac{3}{2})x_0^2)^{1/2} > 0 \tag{5.11}$$

$$d = k^2 + (x_0 - 1)^2, \quad \alpha_\pm = \frac{bV_{m\pm}}{8k_2 - bV_{m\pm}},$$

$$b = 4(x_0 - 1), \tag{5.12a, b, c}$$

$$V_{m\pm} = \tfrac{1}{2}[-b \pm (b^2 + 16k^2)^{1/2}]. \tag{5.12d}$$

$C_2(\omega)$ may be interpreted as a linear transfer function. It is also referred to as scaling actor (Beigie et al., 1991). From the linearity of Eq. (3.15) with respect to the perturbative terms it follows that, if the perturbation is given by Eqs. (5.2) and (5.4), the expression for the Melnikov process is

$$m(s, \theta_{10}, \theta_{20}, \ldots, \theta_{l0})$$
$$= rC_1 + [(1/2\pi)(2/l)]^{1/2}\sigma\sum_{n=1}^{l} C_2(\omega_n)$$
$$\times \cos[\omega_n t + (\omega_n s + \theta_{n0})]. \tag{5.13}$$

The expectation and variance are (Frey and Simiu, 1993):

$$E[m(s, \theta_{10}, \theta_{20}, \ldots, \theta_{l0})] = rC_1. \tag{5.14}$$

$$\mathrm{Var}\,[m(s, \theta_{10}, \theta_{20}, \ldots, \theta_{l0})]$$
$$= \frac{\sigma^2}{2\pi}\int_0^\infty C_2^2(\omega)\Psi_0(\omega)d\omega. \tag{5.15}$$

Example. We consider the case, also studied by Allen et al. (1991), $\delta = 0.3003$, $\tau_0/r = 3.236 \times 10^{-3}$. The unperturbed system has the fixed points $\{0, 0\}$, $\{1.236, 0\}$, and $\{1.764, 0\}$ (Eqs. (5.5)). From Eqs. (5.9) to (5.12), $C_1^+ = 2.524$, $C_1^- = -7.076$, and

$$C_2^+(\omega) = -4.8\sinh(2.064\omega)/\sinh(5.500\omega) \tag{5.16}$$

$$C_2^-(\omega) = -4.8\sinh(3.436\omega)/\sinh(5.500\omega). \tag{5.17}$$

For harmonic forcing (Eq. (5.7)) with $\omega = 1$ (a case examined by Allen et al., 1991), and assumed therein also to correspond to a dimensional time $T_{pk} \approx$ four days), $C_2^+ = -0.152$, $C_2^- = -0.609$, and the necessary condition for exits from the region corresponding to the interior of the left lobe is satisfied for $\sigma/r > 8.22$, where σ is the standard deviation of the harmonic forcing, i.e., $\sigma = \tau_1/(2)^{1/2}$. The inequality $\sigma/r > 11.74$ obtains for the right well. If $\sigma/r < 8.22$ there can be no exits from the left region and, a fortiori, from the right region as well. Note that, for $\omega = 1$, the excitation needed to satisfy the necessary condition for the occurrence of chaos is stronger for the right well (which is the smaller of the two wells) and weaker for the larger well.

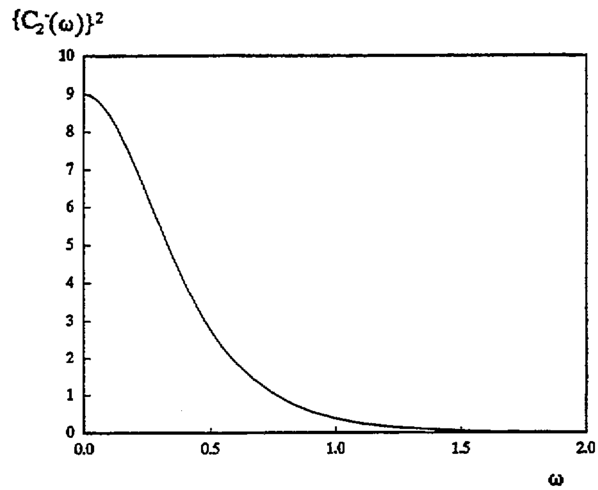

Fig. 2 Square of transfer function, $[C_2^-(\omega)]^2$

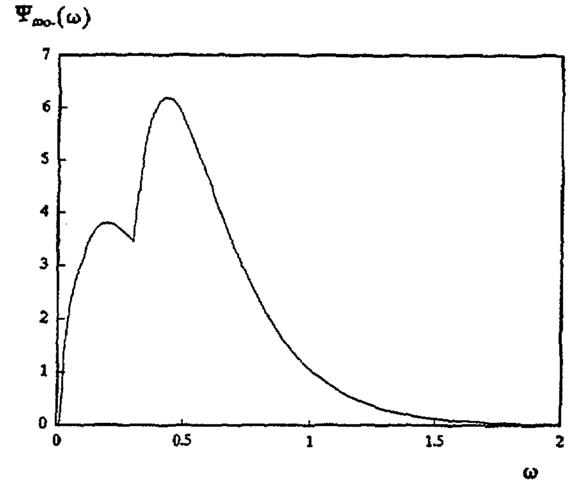

Fig. 3 Spectral density of Melnikov process $\Psi_{mo-}(\omega)$

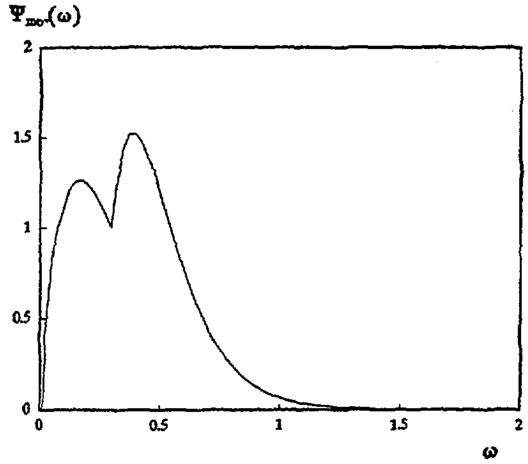

Fig. 5 Spectral density of Melnikov process $\Psi_{mo+}(\omega)$

We now consider the case of random forcing with spectrum $\sigma^2\Psi_0(\omega)$. Figures 2 and 3 show, respectively, the square of the transfer function, $[C_2^-(\omega)]^2$, and the spectral density of the Melnikov process for $\sigma = 1$, $\Psi_{mo-}(\omega) = [C_2^-(\omega)]^2\Psi_0(\omega)$. Figures 4 and 5 represent $[C_2^+(\omega)]^2$ and $\Psi_{mo+}(\omega) = [C_2^+(\omega)]^2\Psi_0(\omega)$, respectively. Equation (4.5) applied to $\Psi_{mo-}(\omega)$ and $\Psi_{mo+}(\omega)$ yields $\nu^- = 0.0702$ and $\nu^+ = 0.0534$, respectively. The transfer functions are seen to suppress or considerably reduce the spectral components of the wind stress with frequencies $\omega > 1$ or so, and to amplify lower frequency components. The standard deviations of the Melnikov process are $\sigma_{m+} = 0.359\sigma$ and $\sigma_{m-} = 0.778\sigma$, respectively, so $k_{-1} = |E[m_-]/\sigma_{m-}| = 9.1r/\sigma$ and $k_+ = |E[m_+]/\sigma_{m+}| = 7.03r/\sigma$. We assume $\sigma/r < 8.22$, say, $\sigma/r = 4$, so $k_- = 2.275$ and $k_+ = 1.76$. From Eq. (4.4b), $E(k_-) = 0.0053$ and $E(k_+) = 0.0113$. We have $1/E(k_-) = 188.6$ (i.e., $188.6 \times 4/(2\pi) = 119.7$ days) and $1/E(k_+) = 88.5$ (56.3 days), so $1/E(k_-) > 1/E(k_+)$, as one would intuitively expect (the left well is larger than the right well). For $T = 1$ month (i.e., 47.1 nondimensional time units), $p_{T-=1mo} = 0.22$ (Eq. (4.4a)); for one year $p_{T-=1yr} = 0.95$. Also, $p_{T+=1mo} = 0.41$, $p_{T+=1yr} = 0.998$.

Suppose a decision would hinge on whether, given the wind spectrum (5.1), the probability of nonoccurrence of chaotic jumps in the current would be at least 0.5 during one month. The lower bounds $1 - p_{T-=1mo} = 1 - 0.22 = 0.78$ and $1 - p_{T+=1mo} = 0.59$ would provide a conservative basis for such a decision.

To recapitulate, the possibility of chaos was assessed using the Melnikov approach for two cases for which the values of δ and τ_0/r were the same. In the first case the excitations induced by wind were harmonic with standard deviation $\sigma = 8.22r$ and frequency $\omega = 1$ (dimensional period T = four days), that is, the entire energy of the wind stresses was concentrated at the frequency of the wind stress spectral peak. From the Melnikov approach it follows that no exits are possible in this case.

In the second case the wind stress fluctuations were random and were derived from the Van der Hoven wind spectrum whose peak is at T_{pk} = four days. The standard deviation of the wind stresses was $\sigma = 4r$, that is, less than half as large as that of the largest harmonic forcing that precludes the occurrence of exits ($\sigma = 8.2r$). In this case the necessary conditions for the occurrence of chaos were satisfied with relatively high probability for an interval of one year, and with probability of the order of about 0.4 for an interval of one month. The probabilities would be considerably higher if the standard deviation of the wind stresses were assumed to be $\sigma = 8.2r$.

Illustrations of chaotic motions for $\delta = 0.3003$, $\epsilon = 0.001$, $\tau_0/r = 3.236$ and $r = 0.01$ are shown in Figs. 6(a) and 6(b). Figures 6 show time histories of $x(t)$ for (a) periodic forcing, $\sigma = 21.21$ ($\tau_1 = 30$), $\omega = 1$, $x(0) = 1.236$, $y(0) = z(0) = 0$, and (b) stochastic forcing (see Appendix), $\sigma = 8$, $x(0) = 10^{-5}$, $y(0) = z(0) = 0$. We note that while theoretical proofs require the use of uniformly bounded and uniformly continuous noise, the method used to generate a noise realization numerically need not conform to this requirement. The time histories were obtained by numerical integration with tolerances 10^{-6}. The sensitivity to initial conditions was verified numerically by following the evolutions in time of small separations introduced

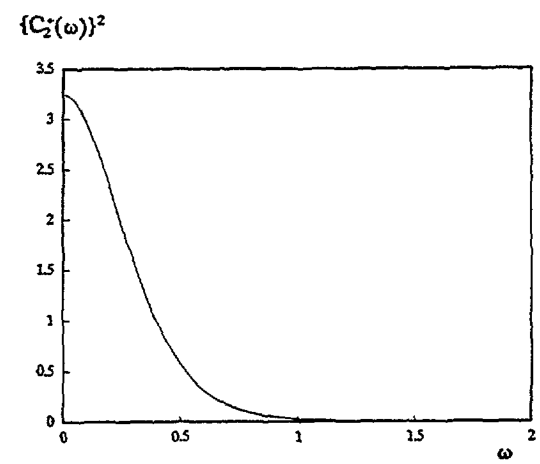

Fig. 4 Square of transfer function, $[C_2^+(\omega)]^2$

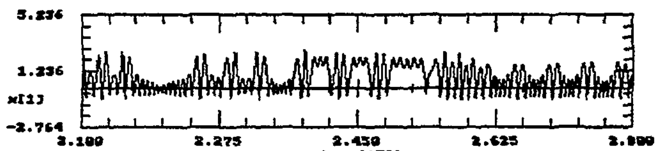

Fig. 6(a)

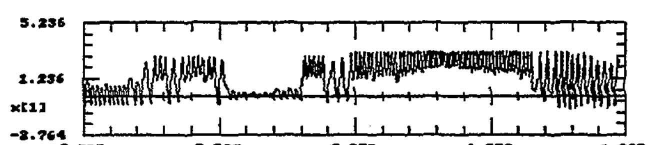

Fig. 6(b)

Fig. 6 Record of chaotic motion for (a) harmonic forcing; (b) realization of random forcing in the initial values of the equations of motion—see, e.g., Bergé, Pomeau, and Vidal, 1984, p. 129.

Summary and Conclusions

For stochastically perturbed systems, the applicability of the Melnikov approach can be extended from a class of two-dimensional oscillators (Frey and Simiu, 1993) to a class of slowly varying oscillators studied by Wiggins and Holmes (1987) and Wiggins and Shaw (1988) for periodic perturbation. This extension was applied to a model of offshore currents driven by wind speed fluctuations, studied by Allen et al. (1991) for the periodic fluctuation case. The following results were obtained: (1) necessary conditions for chaos induced by stochastic perturbations depend on the shapes of both the spectra of the stochastic excitations *and* the transfer functions in the expression for the Melnikov transform; (2) the development of a stochastic Melnikov process allows the estimation of (a) a lower bound for the mean time of exit from each of the regions of phase space associated with a potential well; and (b) provided that the excitation is sufficiently small in relation to the damping, of an upper bound for the probability of occurrence of exits from each well during a specified time interval.

We then applied the Melnikov approach to a model of wind-driven coastal currents. If the wind fluctuations were assumed to be deterministic, the Melnikov approach led to the conclusion that for a certain range of parameters exits could not occur. For the same system, with the same parameters, but assumed to be excited by randomly fluctuating winds, the stochastic Melnikov approach led to the conclusion that the necessary conditions for the occurrence of exits during various specified time intervals were in fact satisfied with relatively high probabilities. Applying a deterministic Melnikov approach to a stochastic system may thus yield the result that exits cannot occur under conditions in which their occurrence is in fact possible.

Acknowledgments

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APPENDIX

For the record of Fig. 6(b), $G(t) = \Sigma\, a_n \cos \omega_n t) + b_n \sin (\omega_n t)$, where $n = 1, 2, \ldots, 8$, $\omega_n = 0.2n$ and a_n, b_n were obtained by simulation (Rice, 1957). The values $\{a_1, b_1; a_2, b_2; \ldots; a_8, b_8\}$ were $\{.27, -.27; -.96, -.87; .84, .48; .33, -.68; .69, .58; .52, .34; .75, -.24; -.09, 1.31\}$.